Design of Fractional Order Sliding Mode Controller for Chaos Suppression of Atomic Force Microscope System

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Abstract: A novel nonlinear fractional order sliding mode controller is proposed to control the chaotic atomic force microscope system in presence of uncertainties and disturbances. In the design of the suggested fractional order controller, conformable fractional order derivative is applied. The stability of the scheme is proved by means of the Lyapunov theory based on conformable fractional order derivative. The simulation results show the advantages of the designed controller such as fast convergence speed, high accuracy and robustness against uncertainties and disturbances.

Keywords: Conformable Fractional Order Derivative, Chaotic System, Atomic Force Microscope System, Fractional Order System, Fractional Order Sliding Mode Controller.

1 Introduction

The atomic force microscope is a known device for imaging the topography of surfaces and surface analysis applications with the precise measurements at the nano-scale [1]. The atomic force microscope is composed of a probe with a microscopic tip joined to a cantilever. The force between probe tip and the sample surface leads to cantilever deviation. Using optical methods, cantilever deflection is calculated. Micro-cantilever has chaotic behavior under in the specific conditions [2]. In [3], the micro-cantilever is modelled. Also, in order to suppress the chaotic behavior of atomic force microscope, a proportional and derivative controller is presented. In [4], a robust feedback controller is developed to control the chaotic behavior of atomic force microscope system. In order to control the chaotic behavior of atomic force microscope system, two control schemes are investigated in [5].

Fractional calculus is known as an effective tool in many applications. So far, several definitions of fractional order derivatives have been presented [6]. Recently, conformable fractional order derivative as a new definition of fractional order derivative is introduced. One of important its advantages is having simple calculation [7]. Various papers have been presented based on conformable fractional order derivatives. In [8], some important laws and definitions based on conformable fractional order derivative are presented. Fractional Newtonian mechanics based on conformable fractional calculus are studied in [9]. Stability of fractional differential systems is discussed using conformable fractional order derivative in [10].

Sliding mode controller is an effective strategy to control systems with uncertainties and disturbances. In [11], two new nonlinear sliding mode controllers are developed. In [12], a chattering-free full-order nonlinear sliding mode controller is developed. In order to control type I diabetes in presence uncertainties and disturbances, a fractional order sliding mode controller and adaptive fractional order sliding mode controller are designed in [13]. For an uncertain manipulator, a fuzzy robust fractional order controller is developed in [14]. Fractional sliding mode schemes are presented to track and stabilize some nonlinear fractional-order systems with uncertainty in [15].

In this study, a fractional order sliding mode controller involving a novel switching function based on conformable definition is designed to remove chaotic behavior of atomic force microscope system. The stability analysis for the proposed controller is discussed.

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using Lyapunov theorem based on conformable operators. The simulation results show the effectiveness of the developed controller.

The paper is organized as follows: Some mathematical preliminaries are presented in Section 2. Mathematical model of atomic force microscope system is introduced in Section 3. In Section 4, fractional order Lyapunov stability based on conformable fractional order derivative is investigated. A novel fractional order sliding mode controllers with conformable fractional order derivative is discussed in Section 5. Section 6 demonstrates simulation results of the suggested scheme. Finally, Section 7 concludes this article.

2 Basic Definitions and Preliminaries

In this section, some basic definitions and preliminaries of fractional calculus are presented.

**Definition 1** [7]. The below fractional order definition is called conformable fractional derivative.

\[ T^\alpha(f)(t) = \lim_{h \to 0} \frac{f(t + \alpha h) - f(t)}{h} \]  \hspace{1cm} \text{(1)}

where \( f \colon [0, \infty) \to \mathbb{R} \) and \( 0 < \alpha < 1 \).

**Definition 2** [7]. The conformable fractional order integral is defined as:

\[ T^{-\alpha}(f)(t) = \int_0^t f(\tau) \tau^{-1-\alpha} \, d\tau \] \hspace{1cm} \text{(2)}

where, \( \alpha \in (0, 1) \).

**Theorem 1** [7]. Consider \( f(t) \) be a continuous function such that \( T^\alpha f(t) \) exists. Then,

\[ T^{-\alpha}(T^\alpha f)(t) = f(t), \quad \text{for} \quad t \geq 0 \] \hspace{1cm} \text{(3)}

where, \( \alpha \in (0, 1) \).

**Definition 3** [8]. The fractional Laplace transform of order \( \alpha \) of \( f(t) \) is defined as:

\[ L^\alpha\{f(t)\}(s) = \int_0^\infty e^{-st} f(t) t^{-\alpha-1} \, dt \] \hspace{1cm} \text{(4)}

where, \( \alpha \in (0, 1] \) and \( f \colon [0, \infty) \to \mathbb{R} \).

**Lemma 1** [8]. Consider \( f \colon \mathbb{R} \to \mathbb{R} \), its fractional Laplace transform is defined as

\[ L^\alpha\{g(t)\}(s) = \int_0^\infty e^{-st} g(t) \, dt \] \hspace{1cm} \text{(5)}

where, \( L\{g(t)\}(s) = \int_0^\infty e^{-st} g(t) \, dt \).

**Theorem 2** [7]. Consider \( \alpha \)-differentiable functions \( f(.) \) and \( g(.) \), some properties of conformable fractional order derivatives are as

1) \( T^\alpha(a + bg) = aT^\alpha(f) + bT^\alpha(g) \), for all \( a, b \in \mathbb{R} \)

2) \( T^\alpha(p^p) = pT^\alpha(p) \) for \( p \in \mathbb{R} \)

3) \( T^\alpha(0) = 0 \), for all constant functions \( f(t) \)

4) \( T^\alpha(f(t) + g(t)) = T^\alpha(f(t)) + T^\alpha(g(t)) \)

5) \( T^\alpha ( \frac{f^\prime}{g} ) = \frac{T^\alpha f}{g} + \frac{T^\alpha g}{g^2} \frac{f^\prime}{g} \)

6) If \( f(.) \) is differentiable, then \( T^\alpha(f)(t) = t^{-\alpha} \frac{df}{dt} \)

where \( \alpha \in (0, 1] \).

In this paper, the notations \( T^\alpha \) and \( T^\alpha \) denote conformable fractional order derivative and integral, respectively.

3 Mathematical Model of Atomic Force Microscope System

The mathematical model of atomic force microscope system in presence of uncertainties and external disturbances is described by:

\[ \begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= f(t) + d(t) + u(t) \\
\end{align*} \] \hspace{1cm} \text{(6)}

where \( f(t) = -a_1 x_1 + \frac{a_2 a_3}{(x_1 + z)^2} + a_3 (a_4 \cos \tau - a_2 x_2) \), \( |d(t)| \leq D \) and \( |A(t)| \leq \Delta \) are external disturbances and system uncertainties, respectively [4].

4 Stability Analysis

In this section, the Lyapunov direct method is investigated by using conformable fractional order derivative.

Consider conformable fractional dynamic system as follows:

\[ T^\alpha(x(t)) = f(t, x(t)) \] \hspace{1cm} \text{(7)}

where, \( f(t, x(t)) \) is a nonlinear function that describes dynamics of the system (7).

**Definition 4.** The system (7) is conformable stable, if its solution satisfies the below inequality:

\[ \|x\| \leq \left\{ h(x(t_0)) e^{-\frac{(t-t_0)}{T}} \right\} \] \hspace{1cm} \text{(8)}

where, \( t_0 \) is the initial time, \( \alpha \in (0, 1) \), \( \lambda > 0 \), \( b > 0 \), \( h(0) = 0 \), \( h(x(t_0)) \geq 0 \). The inequality (8) is the solution of Eq. (7) such that its origin is stable.

In the sequel, without loss of generality, we assume that the equilibrium point is in the origin.

**Theorem 3.** Assume that there exist a Lyapunov function as \( V(t, x(t)) \colon [0, \infty) \times \mathbb{R} \to \mathbb{R} \). If it satisfies (9) and (10), the equilibrium point is conformable stable.

\[ \begin{align*}
l_1 \|x\|^p &\leq V(t, x(t)) \leq l_2 \|x\|^p \\
l_3 \|x\|^p &\leq V(t, x(t)) \leq l_4 \|x\|^p \end{align*} \] \hspace{1cm} \text{(9)}

\[ \begin{align*}
l_3 \|x\|^p &\leq V(t, x(t)) \leq l_4 \|x\|^p \end{align*} \] \hspace{1cm} \text{(10)}
where, $t \geq 0$, $a \geq 0$, $a \in (0,1)$, $l_1$, $l_2$, $l$, $a$ and $b$ are arbitrary positive constants.

**Proof.** By using Ref. [16], according to (9) and (10), we have $T^{-\alpha} V(t, x(t)) \leq \frac{1}{l_1} V(t, x(t))$.

Taking the fractional Laplace transform yields to:

$$sV(s) - V(0) \leq -\frac{1}{l_2} V(s)$$

Thus, we have

$$V(s) \leq \frac{V(0)}{s + \frac{l_1}{l_2}}$$

where $V(0) = V(0, x(0)) \geq 0$.

Applying inverse fractional Laplace transform to the equation (12), $V(t) \leq V(0) e^{-\frac{l_1}{l_2}t}$ is obtained. According to (9) and (10), $\|x\| \leq \left[\frac{V(0)}{l_3} e^{-\frac{l_1}{l_2}t} \right]^{\frac{1}{a}}$ is achieved.

Consider $h(t)$ as follows:

$$h(t) = \frac{V(0)}{l_1} = \frac{V(0, x(0))}{l_1} \geq 0$$

Hence, we have:

$$\|x(t)\| \leq \left[\frac{l_3}{l_1} e^{\frac{l_1}{l_3}t} \right]^{\frac{1}{a}}$$

So, system (7) is conformable stable.

### 5 Design Procedure

In this section, the design procedure of new fractional order sliding-mode controller is presented. The design of the proposed fractional order sliding mode controller involves developing nonlinear fractional order sliding surface with desired system dynamics as well as switching function design.

The sliding surface is designed as:

$$S(t) = \frac{1}{2} \left[ x_2^2(t) + m_1 x_1^2(t) \right]$$

Taking the time derivative of the above equation, one can obtain:

$$T^{-\alpha} S(t) = \left[ b_1 \tanh(c_1 x_1 + c_2 x_2 + c_3 x_1^{15} + c_4 \tau \tau x_1^{15}) \right]^{\beta} + k \tanh^2 \left[ T^{-\alpha} \left( x_1(t) + x_2(t) \right) \right]$$

For obtaining equivalent control law, sliding surface derivative must satisfy the below equation.

$$T^{-\alpha} S(t) = 0$$

In absence of uncertainty and according to system dynamics, the equivalent control is obtained as:

$$u_{eq}(t) = \left[ -m_1 x_1 + f(t) \right] + \left[ b_1 \tanh(c_1 x_1 + c_2 x_2 + c_3 x_1^{15} + c_4 \tau \tau x_1^{15}) \right]^{\beta} - k \tanh^2 \left[ T^{-\alpha} \left( x_1(t) + x_2(t) \right) \right]$$

Afterwards, the reaching law is designed as:

$$u_a(t) = -k \left[ \tan(a_s \tau(s) + a_T \tau^s(s)) \right]$$

where $\tau$ is obtained. According to (9) and (10), we have

$$\tau(t) \leq \left[ \frac{V(0)}{l_4} e^{-\frac{l_1}{l_5}t} \right]^{\frac{1}{a}}$$

Then let us consider the Lyapunov function candidate as:

$$V(t) = \left[ \frac{V(0)}{l_3} e^{-\frac{l_1}{l_3}t} \right]^{\frac{1}{a}}$$

The derivative of $V(t)$ is obtained as:

$$\dot{V}(t) = x_2(t) \left[ f(t) + m_1 x_1(t) + \Delta f(t) + d(t) + u(t) \right] + x_1(t) x_2(t)$$

According to (6) and (20), we have (23).
Simplifying the above equation, results in
\[
v(t) \leq x_1 \left[\Delta f(t) + d(t) - m_1 x_1(t)\right] \\
- \left[ b_1 \tanh(c_1 x_1(t) + c_2 x_2(t) + c_3 x_3(t)) + c_4 x_4(t) \right]^{\alpha} \\
+ k_1 \tanh^2(T^{-\gamma}(x_1(t) + x_2(t))) \\
- k \left[ \tanh(a_1 x_1(t) + a_2 T^{-\gamma} x_1(t) + a_3 T^{-\gamma} x_2(t)) \right] \\
+ m_2 x_1(t) x_2(t) \right] \leq |x_3| \left[|\Delta| + |D| + |b_1| - k_1 + k
\]

For the below condition, \( v(t) \leq 0 \) is achieved.

\[ k \leq k_1 - \left[|\Delta| + |D| + |b_1| \right] \] (24)

This completes the proof.

**Theorem 5.** The state trajectories of the controlled system (6) by the controller (20) converge to the nonlinear fractional order sliding surface \( s = 0 \) in a finite time.

**Proof.** Considering \( V(s) = \frac{1}{2} s^2(t) \), one has

\[
T^\alpha V_{SMC}(t) = S(t) T^\alpha \dot{S}(t) \leq -\eta |s(t)|
\] (25)

Substituting Eq. (16) in Eq. (25) yields,

\[
T^\alpha V_{SMC} = S(t) T^\alpha \dot{S}(t) \\
= S(t) \left[ f(t) + \frac{\Delta f(t)}{s(t)} + d(t) + u(t) + m_1 x_1(t) \right] \\
+ \left[ b_1 \tanh(c_1 x_1(t) + c_2 x_2(t) + c_3 x_3(t)) + c_4 x_4(t) \right]^{\alpha} \\
+ k_1 \tanh^2(T^{-\gamma}(x_1(t) + x_2(t))) \\
- k \left[ \tanh(a_1 x_1(t) + a_2 T^{-\gamma} x_1(t) + a_3 T^{-\gamma} x_2(t)) \right] \\
+ m_2 x_1(t) x_2(t) \right] \leq |x_3| \left[|\Delta| + |D| + |b_1| - k_1 + k
\]

Thus, we have:

\[
T^\alpha V_{SMC}(t) = S(t) T^\alpha \dot{S}(t) \\
= S(t) \left[ f(t) + \frac{\Delta f(t)}{s(t)} + d(t) + u(t) + m_1 x_1 \right] \\
- m_1 x_1 \left[ b_1 \tanh(c_1 x_1(t) + c_2 x_2(t) + c_3 x_3(t)) + c_4 x_4(t) \right]^{\alpha} \\
+ f(t) - k \left[ \tanh(a_1 x_1(t) + a_2 T^{-\gamma} x_1(t) + a_3 T^{-\gamma} x_2(t)) \right] \\
+ k \left[ \tanh(a_1 x_1(t) + a_2 T^{-\gamma} x_1(t) + a_3 T^{-\gamma} x_2(t)) \right] \\
- m_2 x_1(t) x_2(t) \right] \leq |x_3| \left[|\Delta| + |D| + |b_1| - k_1 + k \right]
\]

Simplifying the (29) results in:

\[
T^\alpha V_{SMC}(t) = \\
S(t) \left[ f(t) + \frac{\Delta f(t)}{s(t)} + d(t) + u(t) + m_1 x_1 \right] \\
- k \left[ \tanh(a_1 x_1(t) + a_2 T^{-\gamma} x_1(t) + a_3 T^{-\gamma} x_2(t)) \right] \\
+ k \left[ \tanh(a_1 x_1(t) + a_2 T^{-\gamma} x_1(t) + a_3 T^{-\gamma} x_2(t)) \right] \\
- m_2 x_1(t) x_2(t) \right] \leq |x_3| \left[|\Delta| + |D| + |b_1| - k_1 + k \right]
\]

Then:

\[
|\Delta| + |D| + \eta \leq k \left[ \tanh(a_1 x_1(t) + a_2 T^{-\gamma} x_1(t) + a_3 T^{-\gamma} x_2(t)) \right] \\
|\Delta| + |D| + \eta \leq k
\] (30)

So, for \( |\Delta| + |D| + \eta \leq k \), the proof is completed.

The state trajectories of the controlled system will converge to zero asymptotically. In the following, the convergence to zero in finite time is shown.

According to reaching condition Eq. (25):

\[
T^\alpha V(t) = s(t) T^\alpha s(t) \leq -\eta |s(t)|
\] (31)

where, \( T^\alpha s(t) = t^{-\alpha} ds(t) dt \).  So

\[
\frac{s(t)}{|s(t)|} ds(t) \leq -\eta |s(t)|^{-1} dt
\] (32)

Integration from both sides of the above equation, we have:

\[
\int_{t_0}^{t(t)} \left[ s(t) \right] dt \leq -\eta t \frac{t}{\eta}
\]

\[
\int_{t_0}^{t(t)} \left[ s(t) \right] dt \leq -\eta t \frac{t}{\eta}
\]

\[
t \leq \left[ \frac{\alpha}{\eta} \left( 0 - s(0) \right) \right]^{-\alpha} s(t) > 0
\]

\[
t \leq \left[ \frac{\alpha}{\eta} \left( 0 - s(0) \right) \right]^{-\alpha} s(t) < 0
\] (33)

So, the reaching time is as:

\[
t \leq \left[ \frac{\alpha}{\eta} \left( 0 - s(0) \right) \right]^{-\alpha}
\] (34)
6 Simulation Results

This section is utilized to confirm the performance of the designed scheme. Consider the atomic force microscope system (6) with the follow parameters [4]:

\[ a_1 = \frac{4}{27}, \quad a_2 = 1.2, \quad a_3 = 1, \quad a_4 = -2.9, \quad a_5 = 0.1, \]
\[ z = 2.5, \quad \Delta f = 0.1\sin(4\pi x_1)\sin(4\pi x_2), \quad \text{and} \quad d(t) = 0.2\sin(0.5t). \]

Where, the parameters of controller are gained by manipulating as follow:

\[ m_1 = \frac{4}{27}, \quad b_1 = 1.8, \quad c_1 = 30, \quad c_2 = 10, \quad c_3 = 30, \quad c_4 = 30, \quad k_1 = 4.5, \quad k = 0.4, \quad \alpha = 0.97, \quad \beta = 0.65, \quad q = 0.75. \]

The control input is activated from \( t = 50. \) For the initial value \([x_1(0), x_2(0)] = [0, 0]^T. \)

Fig. 1 demonstrates the state trajectories of the controlled system (6) under control law (20). Fig. 2 shows the control input. The simulation results demonstrate the advantages of the proposed method such as fast convergence and high accuracy. Also, it is obvious that using conformable fractional order operators leads to simple mathematic calculations in control design procedure.

Figs. 3 and 4 show the proposed controller improved the convergence speed comparing with [11] and [12].

Fig. 3 represents the results applying the proposed controller in comparison with the designed controller adopted from [11] with the control law as

\[ u = -f(t) - 7\operatorname{sgn}(x_2)\left[x_2^{\frac{9}{2}} - 10x_1^{\frac{9}{3}} + u_n \right] \]

where \( f(t) = -a_1x_1 + a_2\left( \frac{a_1}{x_1 + z} \right) + a_3(a_4\cos \tau - a_5x_2) \) and \( u_n = 0.\operatorname{sign}(v) \), \( v = -10\operatorname{sign}(s) \).

Fig. 4 shows the state trajectories of the controlled system in comparing with the controller adopted from [12] with the control law as

\[ u = -f(t) - g_2z - 2\varepsilon z^{\frac{1}{5}} - 0.01\frac{s}{|s| + \delta} - 0.2s \]

where \( f(t) = -a_1x_1 + a_2\left( \frac{a_1}{x_1 + z} \right) + a_3(a_4\cos \tau - a_5x_2), \)
\[ g_2 = b_1(-b_1(-b_2z_1 + z_2)), \] and \( b_1 = 0.2. \)

In Fig. 5, control inputs are illustrated.

7 Conclusion

This article introduces the fractional order sliding mode controller with the novel fractional order switching law to stabilize and suppress the chaotic behaviour of atomic force microscope in presence of uncertainty and disturbance. The control scheme is based on conformable fractional order derivative. The developed controller has some advantages such as quick convergence speed, high accuracy and robustness against uncertainties and disturbances. The finite-time stability analysis is performed by using the Lyapunov theory based on conformable fractional order derivative. Finally, simulation results are given to show the efficiency of the proposed scheme.
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Fig. 3 Responses of system state trajectories; a) $x_1$ and b) $x_2$.

Fig. 4 Responses of system state trajectories; a) $x_1$ and b) $x_2$.

Fig. 5 Control inputs.
References


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