Modeling of the Maximum Entropy Problem as an Optimal Control Problem and its Application to pdf Estimation of Electricity Price

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Abstract: This paper proposes a novel two step modeling and analysis on the continuous random variable of electricity price. At the first step, the continuous optimal control theory is used to model and solve the maximum entropy problem for a continuous random variable. The maximum entropy principle provides a method to obtain least-biased Probability Density Function (pdf) estimation. In this paper, to find a closed form solution for the maximum entropy problem with any number of moment constraints, the entropy is considered as a functional measure and the moment constraints are considered as the state equations. Therefore, the pdf estimation problem can be reformulated as the optimal control problem. At the second step, the proposed unbiased pdf estimator is used to estimate the pdf of electricity price. Moreover, the statistical indices and the distributional characteristics of electricity price are analyzed at each load level. The simulation results on the electricity price data of New England, Ontario and Nord Pool electricity markets show the efficiency of the proposed pdf estimator. In addition, the obtained results show that by decreasing the load, the statistical and distributional characteristics of the electricity price inclined toward the statistical properties of the normal distribution.

Keywords: Electricity price, Maximum entropy (ME), Optimal control, Probability density function (pdf).

1 Introduction

1.1 Motivation

In a restructured electricity market, electricity price is the most important signal for all market participants [1] and has different statistical characteristics compared to other markets. Therefore, several studies have been presented specifically on electricity price analysis and modeling. In [2-4] classifications of the considered methods and tools in electricity price modeling are presented.

Generally, analyses on electricity price are aimed at two different goals. The first goal is to present a forecasting model for the price and its volatility. Important tools applied in this area are such as Time series method, neural networks, wavelet transform, fuzzy logic, Weighted Nearest Neighbors (WNN) techniques and hybrid methods. The second goal is to get a good understanding of the electricity price behavior and electricity market operation using statistical and probabilistic approaches. Results that achieved by these analyses can affect the player's strategies and thus economic benefits of them [5]. Moreover, through a right understanding of the electricity price behavior, the market regulator is able to monitor the level of the competitiveness of the market [6].

1.2 Literature Review

1.2.1 Study the pdf of Electricity Price

Application of the electricity price pdf to risk management, evaluating the risk associated with the deviation of actual prices from a particular forecast, options valuation, etc is indispensable [1]. Moreover, the probability distribution of price is used in short-term, [4, 7, 8], mid-term, [9, 10] and long-term [11] modeling and forecasting of the electricity price. Analysis of the pdf of electricity price at different market conditions plays the role of a monitoring tool for the market operator [6]. The relationship between the

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system surplus capacities percent and price pdf has been indicated in [6]. It's statistically shown that at high load, which the supply and demand relationship becomes intense, the distribution of price deviates from the normal distribution. Proposing an efficient analytic method for assigning a pdf to the random variable of the electricity price has become the center of attention of many researchers. The calculation of electricity price pdf with two methods has been discussed in [1]: statistical method and Artificial Neural Network (ANN). In [12] Inverse-Quantile function has been used to fit two special classes of distributions to electricity price. Panagiotelis and Smith in [13], applied multivariate skew t distribution to estimate the pdf of electricity price. An analytical relationship between the system load level, the structure of price formation and the distribution of Locational Marginal Price (LMP) from the planning viewpoint has been proposed in [14].

1.2.2 Entropy and Electricity Market

Entropy is a measure of uncertainty associated with the random variable. It defines the expected value of the random variable information [15]. This concept has been applied for data mining in the field of electricity market price forecasting in [16]. The entropy coefficient is used as a measure of market concentration in [17, 18]. Permutation entropy, topological entropy and the modified permutation entropy are used as the measures of volatility in electricity markets in [19]. The electricity purchase risk is mainly related to the uncertainties of electricity price. In other words risk is arisen from market change or some uncertain events in the future. This means that the risk and entropy have the same essence [20]. In [20] the information entropy has been introduced as risk measure for electricity purchase.

1.2.3 Optimal Control Theory

Optimal control theory is used to find an optimal solution in controlling a dynamic system. It models a process and its constraints as state equations and finds a control law (optimal control policy) for a given system such that a certain optimality criterion is achieved. Starting point to study a process with optimal control is to model the process as a set of differential equations (state space equations) [21, 22]. Optimal control has numerous applications in design of the marine systems, aerospace, robots, industrial processes, power systems, energy management, economic systems, bio-medical models and control of environment systems [21, 23].

1.3 Contributions

This paper proposes a novel two step modeling and analysis on the continuous random variable of electricity price. At the first step, the problem of the pdf estimation of electricity price is analyzed in viewpoint of information theory. The concept of maximizing the information of a random variable has resulted in the use of the maximum entropy method which provides a mean to obtain least-biased pdf estimator [19]. The maximum entropy problem for a continuous random variable, such as electricity price, introduces a functional optimization problem. Thus, in this paper the structure of optimal control problem is utilized to model entropy maximization problem for continuous random variable. Hence, the entropy is considered as a functional and relevant moment constraints are modeled as the state space equations. One of the noteworthy advantages of the proposed approach is the possibility to easily consider any quantity of moment constraints and to reach a closed form solution for the probably density function using optimal control theory. Moreover, modeling the entropy maximization as a state space and an optimal control problem can be itself very beneficial in other researches.

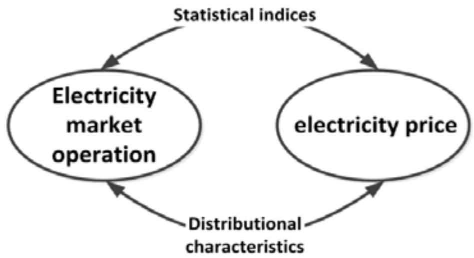

Fig. 1 Market operation and electricity price.

At the second step, the proposed unbiased pdf estimator is used to estimate the probability density functions of the electricity price. Moreover, the statistical indices and the distributional characteristics of electricity price are analyzed at each load level. As shown in Fig. 1, the main goal of these analyses is to provide a relation betwean the electricity price and the electricity market operation.

The simulation results on the electricity price data of New England, Ontario and Nord Pool electricity markets show the efficiency of the proposed pdf estimator. The pdf estimation is done for the electricity price at three different load levels. Moreover, by using the quantitative and qualitative methods, the statistical properties of electricity price are analyzed at each load level. The obtained results show that by decreasing the load, the statistical and distributional characteristics of the electricity price inclined toward the statistical properties of the normal distribution.

1.4 Paper Organization

This paper is organized as follow: the maximum entropy and the optimal control problems are introduced in section 2. The maximum entropy problem is modeled as the optimal control problem in section 3. Section 4 introduces the proposed unbiased pdf estimator for the continuse random variable. Section 5 includes the simulation results of pdf estimation of electricity price. Finally, the paper is summarized and concluded in section 6.

2 Problem Formulation

2.1 Entropy and Maximum Entropy

Entropy of a random variable X, defined by $H(X)$, is a measure of the uncertainty associated with the random variable. Shannon quantifies the entropy of X as the expected value of its information (I). The entropy of discrete random variable X is expressed by:

$$H(X) = E(I) = \sum_{i=1}^{n} pr_i \times I_i = -\sum_{i=1}^{n} pr_i \times \ln pr_i \qquad (1)$$

If X be a continuous random variable, e.g. electricity price signal, the entropy is defined similarly as:

$$H(X) = \int_{x_{\min}}^{x_{\max}} f(x) \ln \frac{1}{f(x)} dx = -\int_{x_{\min}}^{x_{\max}} f(x) \ln f(x) dx \qquad (2)$$

where, $f(x)$ is the pdf of the random variable X and (x_{min}, x_{max}) is the range of variation of X [15].

Maximum entropy method tries to choose a proper probability density function for a random variable based on its statistical moments, such that the maximum information from the data is obtained [15]. In this method, the moment constraints guaranty that the estimated probability distribution is identical to the experimental data. Moreover, considering the concept of information maximization, this method applies minimum presumptions of random variables for estimation of its pdf. In other words, the main advantage of the maximum entropy is to provide a method to obtain least-biased pdf estimation. Hence, it makes sense to term the maximum entropy method for the estimation of the probability density function as an *Unbiased pdf Estimator*. Eq. (3) shows the problem of entropy maximization for a continuous random variable, under the natural constraint and the first m moment constraints.

$$\begin{aligned} &\underset{f(x)}{Max} \quad H = \int f(x) \ln \frac{1}{f(x)} dx = -\int f(x) \ln f(x) dx \\ &such \quad that: \begin{cases} \int_{x_{\min}}^{x_{\max}} f(x) \;\; dx = 1 \\ \int_{x_{\min}}^{x_{\max}} x^r f(x) dx = a_r \qquad r = 1, 2, \ldots, m \end{cases} \end{aligned} \qquad (3)$$

where a_r for r = 1, 2…, m are the constant moment values.

2.2 Optimal Control Problem and the Necessary Conditions for the Optimality

Optimal control is developed to find the optimal control input u^* for a system with state space representation A in Eq. (4), such that the system takes the optimal trajectory y^* in a way that the performance measure J is minimized [22].

$$\begin{aligned} &\dot{y}(t) = A(y(t), u(t), t) \\ &J = h(y(t_f), t_f) + \int_{t_0}^{t_f} g(y(t), u(t), t) dt \end{aligned} \qquad (4)$$

where, t_0 and t_f are the initial and the final times and h and g are the scalar functions. To find optimal control input u^*, at first the Hamiltonian function is formed and then the necessary conditions for optimization are observed:

$$\begin{aligned} &Hamilt(y, u, p, t) = g(y, u, t) + p^T [A(y, u, t)] \\ &\dot{y}(t) = \frac{\partial Hamilt}{\partial p} (y(t)^*, u(t)^*, p(t)^*, t) \\ &\dot{p}(t) = -\frac{\partial Hamilt}{\partial y} (y(t)^*, u(t)^*, p(t)^*, t) \\ &0 = \frac{\partial Hamilt}{\partial u} \left(y(t)^*, u(t)^*, p(t)^*, t \right) \end{aligned} \qquad (5)$$

where $p(t)$ is the Lagrange multipliers vector related to co-state equations.

3 Modeling the Maximum Entropy as an Optimal Control Problem and its Solution

Maximum entropy problem for a continuous random variable such as electricity price introduces a functional optimization. Thus in this section, the Lemma 1 is introduced and proved to model the maximum entropy problem for a continuous random variable as an optimal control problem. Therefore, the entropy is considered as a functional and the moment constraints as the state space equations.

3.1 Modeling the Maximum Entropy as an Optimal Control Problem

By assuming that the x, x_{min} and x_{max} in Eq. (3) are equal with t, t_0 and t_f in optimal control problem in Eq. (4), respectively, and also the decision variable $f(x)$ equals with the control input $u(t)$, then the following lemma is introduced.

Lemma 1:

The maximum entropy problem for a continuous random variable with m moment constraints is equivalent by the optimal control problem with state space representation in Eq. (6) and the performance measure in Eq. (7).

$$\dot{y}(t) = A(y(t), u(t), t) = \begin{bmatrix} u(t) \\ tu(t) \\ \vdots \\ t^m u(t) \end{bmatrix} \qquad (6)$$

$$J = \int_{t_0}^{t_f} u(t) \ln(u(t)) dt \qquad (7)$$

Proof:

Eq. (8) shows that the natural constraint in Eq. (3) could be rewritten as the state space equation with known final condition.

$$\int_{x_{\min}}^{x_{\max}} f(x)\ dx = 1 \xrightarrow[x_{\min}=t_0,\ x_{\max}=t_f]{f(x)=u(t)} \begin{cases} \int_{t_0}^{t} u(\tau)d\tau = y_1(t) \\ y_1(t_f) = 1 \end{cases} \rightarrow \begin{cases} \dot{y}_1(t) = u(t) \\ y(t_f) = 1 \end{cases} \quad (8)$$

Simultaneously, the moment constraints can be transformed. Eq. (9) shows the state space equations which are equivalent by the moment constraints:

$$\int_{x_{\min}}^{x_{\max}} x^r f(x)dx = a_r \rightarrow \begin{cases} \int_{t_0}^{t} \tau^r u(\tau)d\tau = y_r(t) \\ y_r(t_f) = a_r \end{cases} \rightarrow \begin{cases} \dot{y}_r(t) = t^r u(t) \\ y(t_f) = a_r \end{cases} \quad \forall r = 1,\dots,m \quad (9)$$

The state space model in Eq. (6) is obtained from the Eqs. (8) and (9). Moreover, the maximization problem in Eq. (3) can be replaced by the minimization problem in Eq. (10).

$$\underset{f(x)}{Max} \quad H = -\int_{x_{\min}}^{x_{\max}} f(x)\ln f(x)\,dx \quad \Updownarrow_{t_0 = x_{\min}\ \&\ t_f = x_{\max}}^{u(t)=f(x)} \Updownarrow \quad \underset{u(t)}{\min} \quad J = \int_{t_0}^{t_f} u(t)\ln(u(t))dt \quad (10)$$

Therefore, the performance measure in Eq. (7) is obtained from the Eq. (10) and the Lemma 1 is proved.

3.2 Solution of the Equivalent Optimal Control Problem

To find the optimal control input $u^*(t)$ for the system with state space representation in Eq. (6), such that J in Eq. (7) is minimized, first the Hamiltonian function is formed. The Hamilton function related to the equivalent optimal control problem is expressed in Eq. (11).

$$Hamilt(y(t),u(t),p(t),t) = u(t)\ln(u(t)) + \begin{bmatrix} p_1(t) \\ p_2(t) \\ \vdots \\ p_{m+1}(t) \end{bmatrix}^T \begin{bmatrix} u(t) \\ tu(t) \\ \vdots \\ t^m u(t) \end{bmatrix} \quad (11)$$

The necessary conditions for the optimality are presented in Eq. (12). The second necessary condition is the co-state equations. According to Eq. (11), the Hamiltonian function is independent from the state variable y. Therefore, as indicated in Eq. (12), the Lagrange multipliers p_i are constant values. By solving the third necessary condition and considering constant values for the p_i, the optimal control input $u^*(t)$ is obtained.

$$\begin{aligned} \dot{y}(t) &= \frac{\partial Hamilt}{\partial p}(y(t)^*,u(t)^*,p(t)^*,t) = A(y(t)^*,u(t)^*,t) \\ \dot{p}(t) &= -\frac{\partial Hamilt}{\partial y}(y(t)^*,u(t)^*,p(t)^*,t) = 0 \Rightarrow p(t) = cons \\ 0 &= \frac{\partial Hamilt}{\partial u}\left(y(t)^*,u(t)^*,p(t)^*,t\right) \\ &\Rightarrow \ln(u(t)) + 1 + p_1 + tp_2 + \dots + t^r p_{m+1} = 0 \end{aligned} \quad (12)$$

Therefore, the optimal control input $u^*(t)$ can be expressed as Eq. (13).

$$u(t) = \exp(-(1 + p_1 + tp_2 + \dots + t^m p_{m+1})) \quad (13)$$

In Eq. (13) the m+1 Lagrange multipliers p_i are constant values and unknown parameters. These parameters are calculated by using the final conditions in Eqs. (8) and (9). Therefore, the Lagrange multipliers p_is are obtained by solving the m+1 simultaneous equations in Eq. (14):

$$\begin{aligned} &\int_{t_0}^{t_f} u(t)dt = 1 \\ &\int_{t_0}^{t_f} t^r u(t)dt = a_r \qquad \forall r = 1,2,\dots,m \end{aligned} \quad (14)$$

Because of the complexity of these simultaneous equations, heuristic methods such as genetic algorithm are used to solve the equations.

4 Unbiased pdf Estimation of a Continues Random Variable

As mentioned in section 2, the maximum entropy method can be used as an Unbiased pdf Estimator. In Lemma 1 it was proved that the maximum entropy problem can be reformulated as an optimal control problem. Eqs. (13) and (14) show the solution of this optimal control problem. Based on the proposed method, the following process is presented as an Unbiased pdf Estimator. This process assigns an unbiased probability density function to a continuous random variable (especially electricity price):

1. Omitting the improper data from the existing data of the continuous random variable;
2. Calculating the first statistical moments (a_r) from the modified data;
3. Solving the simultaneous in Eq. (14) and calculating the Lagrange coefficients (p_i).
4. Assigning the unbiased probability density function $f(x) = exp(-(1+p_1+xp_2+\dots+xp_{m+1}))$ to the continuous random variable X. This function is assigned on the interval (x_{min}, x_{max}).

5 Pdf Estimation of Electricity Price and Electricity Market Analysis

One of the main goals of this paper is to estimate the probability distribution of the electricity price and

Table 1 Statistical study of price in the New England, Ontario & Nord Pool electricity markets.

Market	Year	Load level	μ[1]	σ[2]	Sk[3]	Ku[4]
New England	2000	Low	25.09	11.46	1.00	10.77
		Med	40.78	15.33	4.09	69.37
		High	76.97	328.29	17.3	309.1
	2001	Low	22.59	11.14	-0.30	3.93
		Med	39.56	18.37	7.14	120.9
		High	66.89	106.34	8.17	71.4
	2002	Low	24.21	8.47	-0.93	4.6
		Med	35.29	10.48	0.84	5.37
		High	55.28	59.78	12.65	189.6
Ontario	2007	Low	23.87	9.37	-0.10	4.82
		Med	47.44	19.9	1.79	12.06
		High	75.67	24.45	2.57	35.82
	2008	Low	22.46	16.16	0.11	2.48
		Med	49.02	23.00	3.72	41.57
		High	79.03	36.42	2.85	28.36
Nord Pool	2010	Low	37.61	11.54	-1.63	5.074
		Med	51.554	13.24	6.812	109.70
		High	89.23	91.763	9.566	114.703
	2011	Low	35.62	12.52	-0.95	3.187
		Med	48.58	11.498	0.094	5.565
		High	64.912	10.89	-0.41	4.58
	2012	Low	19.189	8.097	-0.20	1.92
		Med	34.35	13.43	1.787	17.57
		High	51.189	27.027	3.789	19.99
Normal distribution			-	-	0	3

[1]Mean, [2]Standard Deviation, [3]Skewness, [4]Kurtosis

Table 2 Lagrange coefficients of price in the New England, Ontario & Nord Pool electricity markets.

Market	Year	Load level	p_1	p_2	p_3	p_4	p_5	Price interval ($/MWh)
New England	2000	Low	-0.03	-5.91	2.72	0.76	8.10	(-2.6, 63.6)
		Med	-0.36	-6.63	0.39	8.55	18.45	(2.2, 120.4)
		High	-1.49	-1.66	1.95	3.15	6.5	(17.8, 128)
	2001	Low	0.20	-8.98	4.80	7.75	7.17	(-6.45, 72.31)
		Med	-0.35	-6.89	1.02	7.81	19.82	(0, 119.9)
		High	-1.2	-2.05	0.08	3.81	6.34	(20.38, 120)
	2002	Low	1.94	-9.48	0.15	2.28	9.73	(-5.96, 50.6)
		Med	0.31	-10.2	3.56	11.96	14.77	(0, 102.4)
		High	-0.64	-6.27	7.01	4.52	1.79	(19.17, 106.4)
Ontario	2007	Low	-0.54	-5.57	0.06	8.30	13.91	(-0.4, 73.94)
		Med	-0.95	-4.09	1.7	5.62	9.48	(4.8, 142.3)
		High	-0.64	-2.80	-1.16	3.82	4.98	(30.58,141.09)
	2008	Low	1.38	-8.46	-1.72	4.01	12.58	(-34, 83.01)
		Med	-0.33	-6.21	0.99	5.85	14.89	(-2.62, 146.1)
		High	-1.21	0.25	-1.66	0.028	3.92	(26.31,146.29)
Nord Pool	2010	Low	1.81	-5.38	-5.24	4.15	5.30	(0.28, 59.58)
		Med	-0.19	-10.1	9.06	8.17	14.94	(8.33, 148.01)
		High	-1.8	0.07	3.44	-0.76	1.90	(40.16,169.25)
	2011	Low	1.04	-4.48	-2.53	5.63	-0.58	(0.36, 55.44)
		Med	-0.27	-7.46	0.5165	10.5	21.12	(3.49, 143.97)
		High	0.07	-6.39	2.41	-1.34	14.43	(36.47,106.89)
	2012	Low	-0.69	-1.12	-0.87	-1.15	4.58	(3.92, 36.23)
		Med	-1.06	-4.16	2.27	4.24	15.66	(6.33, 103.48)
		High	-2.23	2.21	2.43	-1.10	2.50	(28.8, 105)

analysis of the statistical indices and the distributional characteristics of electricity price at the different load levels. Therefore, the proposed nonparametric method, which is an unbiased pdf estimator, is used to estimate the pdf of electricity price. The first four statistical moments e.g. mean, variance, skewness and kurtosis are used as the moment constraints. Skewness and kurtosis are the measures of the "asymmetry" and "peakedness" of the probability distribution of a real-valued random variable respectively. Generally, to analyze the behavior and distributional characteristics of a random variable, the first four moments are used. In [24], it is mentioned that the first four statistical moments are the characterizing moments and the 5th moment and above have no essential affect on the distribution function. Therefore, the proposed unbiased pdf estimator gives $p_1,..., p_5$.

5.1 Statistical Study

The load and price data in the New England, Ontario and Nord Pool electricity markets are used for the statistical study [25-27]. Table 1 shows the results of the statistical study on the electricity price in the considered markets at three load levels. The load levels are the low, median and high load levels. The high load data are the loads that higher than $\mu_l + \sigma_l$, in which μ_l is the mean load and σ_l is the standard deviation of the load. The low load data are the loads that lower than $\mu_l - \sigma_l$. Other load data are considered as the median load data. As shown in this table, the skewness and kurtosis of the electricity price are close to the statistical moments of the normal distribution at low load level (sk=0 and ku=3).

At high load, the price spikes influence the statistical properties of electricity price. Price spikes usually occur at high load and on specific market and network operational conditions. Therefore, price spikes are eliminated from the price data and then the electricity price data are analyzed at each load level. Afterwards the pdf of electricity price at each load level are estimated by using the unbiased pdf estimator, which was proposed in the previous section.

5.2 Unbiased pdf Estimation of Electricity Price

In this subsection, based on the four step process which proposed in section 5, an unbiased pdf is estimated and assigned for the random variable of electricity price at each load level. Table 1 shows the first four moment of electricity price at each load level. By solving the simultaneous in Eq. (14), the Lagrange coefficients $p_1,..., p_5$ are calculated. Table 2 shows the values of $p_1,..., p_5$ for the random variable of electricity price at each market, at each year and at each load level. As noted, these functions are defined in the interval of minimum up to maximum price for each load level.

Table 3 shows the closed form of the unbiased pdf of electricity price at each load level in the New England electricity market at year 2000. As illustrated in this table, the close forms of the pdf are derived based on the Lagrange coefficients which calculated in Table 2. These functions are plotted in Fig. 2.

The proposed unbiased pdf estimator is actually a nonparametric method. Hence, it is expected that the

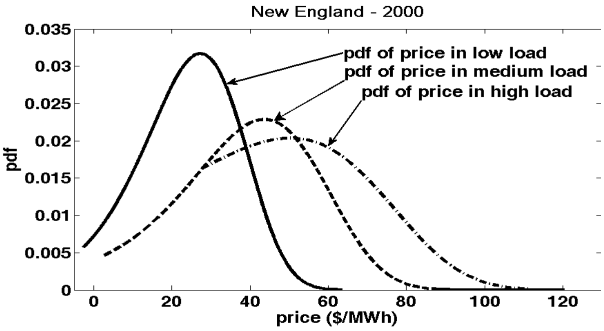

Fig. 2 Unbiased pdf of electricity price at New England electricity market, year 2000.

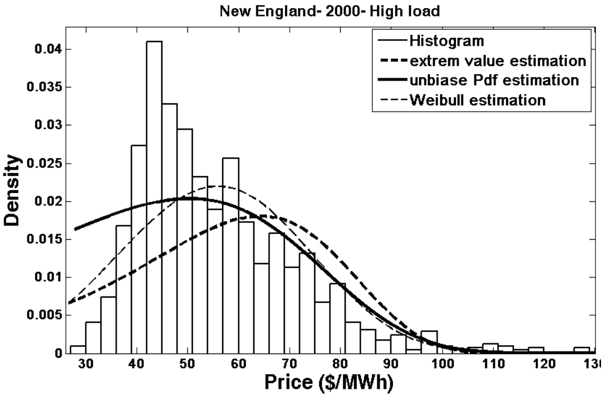

Fig. 3 Comparison of the histogram of electricity price with its estimated probability density functions.

Table 3 Unbiased pdf of electricity price in the New England electricity market at year 2000.

Load	Closed form of pdf of electricity price	Price interval
Low	$f(x)=e^{-(1-0.03-5.91x+2.72x^2+0.76x^3+8.10x^4)}$	(-2.6, 63.6)
Mid	$f(x)=e^{-(1-0.36-6.63x+0.39x^2+8.55x^3+18.45x^4)}$	(2.2, 120.4)
High	$f(x)=e^{-(1-1.49-1.66x+1.95x^2+3.15x^3+6.5x^4)}$	(17.8, 128)

estimated distribution function be more similar to the actual price distribution compared with the other parametric pdf estimators. For instance, in Fig. 3, the histogram of electricity price at high load level is compared with its estimated probability density functions. This figure graphically shows the efficiency of the proposed unbiased pdf estimator compared with Weibull and Extreme Value pdf estimators.

5.3 Electricity Market Analysis

As mentioned in section 1, one of the main goals of this paper is to analyze the statistical behaviors and the distributional characteristics of electricity price at various conditions of the load. In order to achieve this goal, the quantitative and qualitative analyses are used in this subsection.

Table 4 Relative entropy of price in the New England, Ontario & Nord Pool electricity markets.

Market	Year	R. E.[1]	
		Low	High
New England	2001	1.15	5.62
	2002	0.63	5.93
Ontario	2007	1.98	4.92
	2008	0.59	4.28
Nord Pool	2010	1.11	5.97
	2011	0.8	0.76
	2012	0.92	3.73
Normal pdf		0	

[1]Relative Entropy with normal distribution

5.3.1 Quantitative Comparison

The skewness, kurtosis and relative entropy of electricity price are used to quantitative comparison of price at each load level. As illustrated in Table 1 by increasing network load the electricity price skewness moves to the right and kurtosis level elevates. Hence when load increases in these networks, the electricity price pdf deviates from the normal distribution. The relative entropy can be used for the better comparison. Relative entropy is a measure of distance between two distributions. The relative entropy of distribution f and normal distribution φ is defined as bellow [15]:

$$D=\int_{-\infty}^{+\infty} f(x)\times log(\frac{f(x)}{\varphi(x)})dx \tag{15}$$

Indeed, the decrease of the relative entropy to zero is equivalent to the convergence of distribution f to the normal distribution. Table 4 shows the relative entropy of electricity price at low and high load level. As shown in Table 4 by increasing the load the relative entropy of electricity price increases. Hence, when the load increases in these markets, the electricity price pdf deviates from the normal distribution.

5.3.2 Qualitative Comparison

Fig. 4 shows the pdf of electricity price at each load level in New England electricity market in 2000. Furthermore, the normal distributions related to each pdf are shown in this figure. As shown in Fig. 4, by increase of the system load the price distribution deviates from the normal distribution.

Figures 5 and 6 show the q-q plots of electricity price at high and low system load at Nord Pool electricity market at 2012, to compare the probability distributions of them by normal distribution.

The quantile-quantile plot, or q-q plot, is the graphical testing of the equality of two distributions. As shown in Fig. 5, the probability distribution of electricity price at off-peak system load similar to the normal distribution. However, Fig. 6 illustrates that at peak system load, the probability distribution of electricity price is different from the normal distribution.

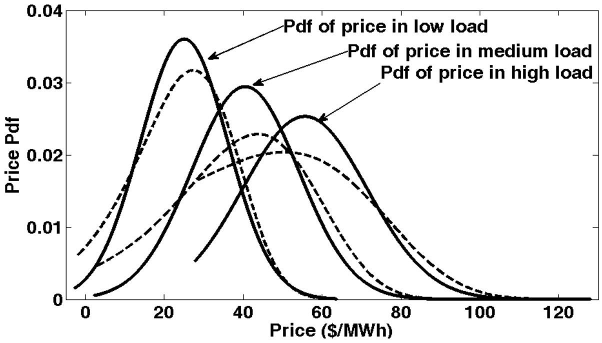

Fig. 4 Comparison of the estimated distribution functions of price with normal distribution.

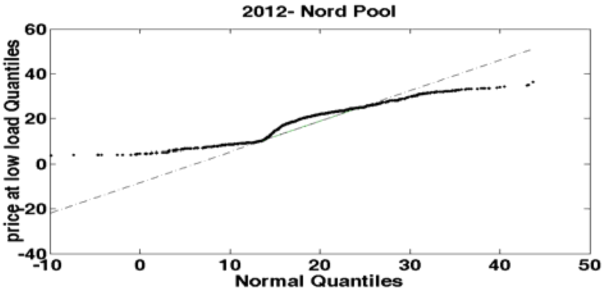

Fig. 5 q-q plot: Compares the probability distribution of price at low load level with normal distribution.

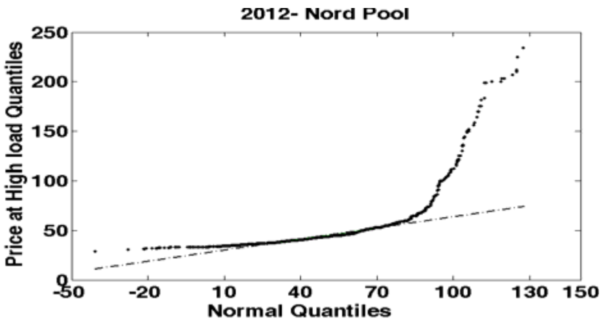

Fig. 6 q-q plot: Compares the probability distribution of price at high load level with normal distribution.

6 Conclusion

The main goal of this paper was to provide a novel two step modeling and analysis on the continuous random variable of electricity price. For this purpose, at the first step the continuous optimal control theory was used to model and solve the maximum entropy problem for a continuous random variable. The maximum entropy principle provides a method to obtain least-biased pdf estimation. In the proposed method, the entropy was considered as a functional measure and the moment constraints were modeled as the state equations to find a closed form solution for the maximum entropy problem with any number of moment constraints. Therefore, the least-biased pdf estimation problem was formulated as an optimal control problem. At the second step, the proposed unbiased pdf estimator was used to estimate the pdf of electricity price. Moreover, the statistical indices and the distributional characteristics of electricity price were analyzed at each load level. The simulation results on the electricity price data of New England, Ontario and Nord Pool electricity markets showed the efficiency of the proposed pdf estimator. The obtained results showed that by decreasing the load, the statistical and distributional characteristics of the electricity price inclined toward the statistical properties of the normal distribution.

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