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Noise Covariance Matrix Estimation in Target Tracking in Three Approaches: n-step Prediction, Kalman Gain Covariance and Gamma-distribution of Noise Statistics

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Abstract: Noise parameters in many target tracking projects are assumed as known factors which is a main challenge because of uncertainty in measurement and state-model noise. Thus, many papers are focused on the accurate estimation of noise statistics. This paper is concentrated on this subject where it is tried to present three simple efficient methods in this regard. Estimation using n-step prediction, applying Kalman filter covariance and using Gamma distribution for noise parameters are the main concepts of the three proposed methods. Simulation results show the efficiency of all methods compared to other methods in the literature where the Gamma-distribution-based method is the most efficient work among other suggested ones in term of estimation error.

Keywords: Noise Covariance Matrix Estimation, Target Tracking, n-step Prediction, Kalman Gain Covariance, Gamma-distribution.

1 Introduction

KNOWLEDGE of dynamic model in each control system is a main challenge of linear and non-linear control engineering problems where existence of noise uncertainty can degrade the performance of optimal control [1]. Noise statistics are unknown in many dynamic-based modeling of the control projects such as target tracking, navigation, signal processing and adaptive control. Many papers are focused on the estimation of noise covariance matrix of model and measurement in recent years [2]. The accurate dynamic modeling needs both matrix of state-space model which is the deterministic part of the model and knowledge of noise statistic which is the uncertain part of the model. False estimation of noise may degrade the optimal control and causes failure on the system performance. Therefore, due to the importance of the noise uncertainty analysis, many papers are devoted to this challenge where some global classifications can be done in this field. First class of methods is based on Bayesian estimation method [3, 4]. In [3], the posterior densities are propagated in time using a predictor method and two moments are propagated in time based on state and observation moment prediction approach. The latest observation is used to update state moments. In [4], Bayesian method is used to achieve the recursive estimation of the noise parameters of the dynamic state and the time-varying measurement in linear state-space models. The proposed approach leads to an adaptive Kalman filter and approximation of the posterior distribution of states and noise of the model. The second class is focused on correlation methods [5-11]. The minimum output sum of squared error (MOSSE) is proposed in [5] to apply the correlation filter of the target tracking. This is the fast tracking method but its accuracy is not guaranteed. Henriques et al. in [6] proposed the circulant structure tracking with kernels (CSK) where the Kernels correlation filter (KCF) in [7] used the feature of multi-channel for tracking. Some improved tracking methods based on correlation are proposed in [8-11]. Maximum likelihood is the other considered method for estimation of noise

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statistics proposed in [12]. Tracking in passive sonar using maximum likelihood estimation is proposed in [13]. Non-linear-based problem with unknown variance and Gauss-Newton type algorithm using the first-order derivatives of the model function is applied in this paper. A modified probabilistic data association is presented for target tracking with assumption of measurement uncertainty and maximum likelihood in [14]. Multi-target tracking using the likelihood of target measurement data is applied in [15] where Integer Linear Programming problem is proposed to be solved. Covariance matching method is the fourth class discussed in [16]. Bearings-only target tracking is applied using the adjustment of the covariance matrix of the filter. Mutual effect of the covariance of the target range estimation and the Cramer-Rao lower error bound is discussed and the stability of a Lyapunov function is employed for analysis of the divergence of the recursive filter [17]. The covariance tracker for target tracking is used in [18] based on covariance matching in consecutive frames and global searching. However, computational complexity is high. Also, some works of this field are concentrated on using Kalman filter [19]. The global approach for linear or non-linear system model based on auto covariance least squares (ALS) methods is proposed to achieve the efficient estimation [20-23]. ALS-based method may degrade in lower accuracy compared to the maximum likelihood method. However, the low complexity of ALS-based method is notable in this approach.

As mentioned in above, finding an efficient method with low complexity for noise parameter estimation is a main challenge in the target tracking field. In this paper, the goal is suggesting simple efficient methods in this regard. Three methods are proposed in this paper where the first method is based on using several predictions due to the measurement data. Simple recursive algorithm is presented in this approach for noise statistics estimation. In the second approach, Kalman gain covariance is used for this goal and in the third method, probability density function is assumed for the noise statistics and based on that, an improved method is achieved. The main contributions of this paper is summarized as follow:

- Presenting a simple recursive noise covariance estimation in the first method based on LS method (method 1: n-step prediction)
- Presenting Kalman-based noise covariance estimation method in the second method (method 2: Kalman gain covariance)
- Presenting a distribution-based approach to achieve an efficient recursive method (method 3: Gamma-distribution noise estimation)
- Comparison of three methods with each other and other effective methods in the literature

The remainder of the paper is organized as follow. In section 2, the system and state-space model is discussed where in section 3, three proposed methods for noise parameter estimation are presented. In section 4, a brief discussion is stated on the advantage and disadvantage of the proposed methods and in section 5, simulation results are presented to show the performance of the proposed methods. Finally, some conclusions are drawn in conclusion section, section 6.

2 System model

Discrete-time linear state-space model is considered in this section which includes state and measurement equations. State equation is given as:

$$\boldsymbol{x}_{k+1} = \boldsymbol{F}_k \boldsymbol{x}_k + \boldsymbol{G}_k \boldsymbol{u}_k + \boldsymbol{w}_k \quad k = 0,1,2,\dots,N \tag{1}$$

where the vector $\boldsymbol{x}_k \in R^{n_x}$ and $\boldsymbol{u}_k \in R^{n_u}$ represent state and control in time instant *k*, respectively. Also, the vector $\boldsymbol{w}_k \in R^{n_x}$ is the noise model which has normal distribution with zero mean and unknown covariance matrix $\boldsymbol{Q} \in R^{n_x \times n_x}$. The measurement equation is described as:

$$\boldsymbol{z}_k = \boldsymbol{H}_k \boldsymbol{x}_k + \boldsymbol{v}_k \tag{2}$$

where the vector $\boldsymbol{z}_k \in R^{n_z}$ and $\boldsymbol{v}_k \in R^{n_z}$ is the measurement noise with normal distribution including zero mean and covariance matrix which may be known or unknown as:

$$cov[\boldsymbol{v}_k \boldsymbol{v}_l] = \delta_{k,l} \boldsymbol{R}_{k,l} \tag{3}$$

where $\delta_{k,l}$ denotes the kronecker delta function. Notice that the matrix $\boldsymbol{F}_k \in R^{n_x \times n_x}$ and $\boldsymbol{G}_k \in R^{n_x \times n_u}$ are the known matrices of the system model and $\boldsymbol{H}_k \in R^{n_z \times n_x}$ is the known measurement matrix. The state noise is assumed to be independent of measurement noise and also of the initial condition of the state. Considering the system model description, the noise covariance matrices of state and measurement are unknown and the purpose of this paper is to estimate these matrices using the known information of state and control matrices of model and measurement described in (1) and (2). In the next section, three methods are proposed to achieve this goal.

3 Proposed Methods

3.1 Method 1: n-step Prediction

In the first method, n-step prediction is assumed using the measurement data and estimation is done based on the difference between the measurement and prediction data as:

$$\boldsymbol{e}_k^n = \boldsymbol{z}_k - \hat{\boldsymbol{z}}_k^n \tag{4}$$

where $\boldsymbol{e}_k^n$ (vector with size of $1 \times k$) represents the error between the n-step previous prediction and measurement. Therefore, we have:

$$\hat{\boldsymbol{z}}_k^n = \boldsymbol{H}_k \prod_{i=1}^{n} \boldsymbol{F}_{k-i} (\boldsymbol{H}_{k-n})^* (\boldsymbol{H}_{k-n}\boldsymbol{x}_{k-n} + \boldsymbol{v}_{k-n}) \quad (5)$$

where globally, $\boldsymbol{A}^*$ is the Hermitian of $\boldsymbol{A}$. Note that $\boldsymbol{F}_k \in R^{n_x \times n_x}$ and $\boldsymbol{G}_k \in R^{n_x \times n_u}$ are the known matrices of the system model and $\boldsymbol{H}_k \in R^{n_z \times n_x}$ is the known measurement matrix. Eq. (5) is derived based on the recursive relation using Eq. (1) and (2) by substituting of each step to the previous prediction one as described for the first step in below:

$$\hat{\boldsymbol{z}}_k^1 = \boldsymbol{H}_k(\boldsymbol{F}_{k-1}\boldsymbol{x}_{k-1} + \boldsymbol{v}_{k-1}) \quad (6)$$

Indeed, the other terms of above equation from substituting are assumed as a total noise which are shown as $\boldsymbol{v}_{k-n}$. (the terms with the form of $\boldsymbol{F}_{k-1}\boldsymbol{w}_{k-1}$). Thus, the error in (4) can be defined as:

$$\begin{aligned}\boldsymbol{e}_k^n &= \boldsymbol{H}_k\boldsymbol{x}_k + \boldsymbol{v}_k - \boldsymbol{H}_k \prod_{i=1}^{n} \boldsymbol{F}_{k-i} (\boldsymbol{H}_{k-n})^* (\boldsymbol{H}_{k-n}\boldsymbol{x}_{k-n} + \boldsymbol{v}_{k-n}) \\ &= \boldsymbol{H}_k\boldsymbol{x}_k + \boldsymbol{v}_k - \boldsymbol{H}_k \left(\prod_{i=1}^{n} \boldsymbol{F}_{k-i}\right) \boldsymbol{x}_{k-n} - \boldsymbol{H}_k^n \boldsymbol{v}_{k-n}\end{aligned} \quad (7)$$

where

$$\boldsymbol{H}_k^n \triangleq \boldsymbol{H}_k \prod_{i=1}^{n} \boldsymbol{F}_{k-i} (\boldsymbol{H}_{k-n})^* \quad (8)$$

Thus, after simplifying the above equation (7), we have:

$$\boldsymbol{e}_k^n = \boldsymbol{v}_k + \sum_{i=0}^{n-1} \boldsymbol{\Omega}_k^i \boldsymbol{w}_{k-i-1} + \sum_{i=0}^{n-1} \boldsymbol{\Omega}_k^i \boldsymbol{G}_{k-i}\boldsymbol{u}_{k-i-1} - \boldsymbol{H}_k^n \boldsymbol{v}_{k-n} \quad (9)$$

where

$$\boldsymbol{\Omega}_k^i \triangleq \boldsymbol{H}_k \prod_{j=1}^{i} \boldsymbol{F}_{k-j} \quad (10)$$

Now, assessing the obtained error in (9), its distribution is normal with mean and covariance matrices as below (knowing that the mean of noise of $\boldsymbol{v}$ and $\boldsymbol{w}$ are zero):

$$E(\boldsymbol{e}_k^n) = \sum_{i=0}^{n-1} \boldsymbol{\Omega}_k^i \boldsymbol{G}_{k-i}\boldsymbol{u}_{k-i-1} \quad (11)$$

and

$$\begin{aligned}\boldsymbol{C}_{k,k'}^n &= E\left(\tilde{\boldsymbol{e}}_k^n (\tilde{\boldsymbol{e}}_{k-k'}^n)^T\right) \\ &= \boldsymbol{\alpha}\delta(\Delta) + \sum_{j=1}^{n-1} \boldsymbol{\beta}_j\, \delta(\Delta - \mathrm{j}) - \boldsymbol{H}_k^n \boldsymbol{R}\delta(\Delta - \mathrm{n}) \\ &\Delta = k - k'\end{aligned} \quad (12)$$

where we have:

$$\boldsymbol{\alpha} = \sum_{i=0}^{n-1} (\boldsymbol{\Omega}_k^i \boldsymbol{Q} (\boldsymbol{\Omega}_k^i)^T + \boldsymbol{R} + \boldsymbol{H}_k^n \boldsymbol{R} (\boldsymbol{H}_k^n)^T) \quad (13)$$

and

$$\boldsymbol{\beta} = \sum_{i=0}^{n-\Delta-1} (\boldsymbol{\Omega}_k^{i+\Delta} \boldsymbol{Q} (\boldsymbol{\Omega}_{k-\Delta}^i)^T) \quad (14)$$

where

$$\tilde{\boldsymbol{e}}_k^n = \boldsymbol{e}_k^n - E(\boldsymbol{e}_k^n) \quad (15)$$

Now, we can write the global linear from and use LS (least square) method to solve the unknown vector of covariance matrices [25, 26]. Based on the LS method, the linear form for estimation due to above equations (13, 14) is written as:

$$\boldsymbol{\theta}_k \boldsymbol{W}_k = \boldsymbol{d}_k \quad (16)$$

where $\boldsymbol{\theta}_k$ and $\boldsymbol{d}_k$ are the input and the desired output of LS estimation process, respectively defined in the matrix notation in the sequel. $\boldsymbol{W}_k$ is the unknown parameter vector of noise covariance matrices ($\boldsymbol{Q}/\boldsymbol{R}$) as:

$$\boldsymbol{W}_k = [\boldsymbol{Q}^T \quad \boldsymbol{R}^T]^T \quad (17)$$

and the matrix $\boldsymbol{\theta}_k$ can be written as:

$$\boldsymbol{\theta}_k = \begin{bmatrix} \sum_{i=0}^{n-1} \boldsymbol{\Omega}_k^i \otimes \boldsymbol{\Omega}_k^i & \boldsymbol{H}_k^n \otimes \boldsymbol{H}_k^n + \boldsymbol{I} \\ \sum_{i=0}^{n-2} \boldsymbol{\Omega}_{k-1}^i \otimes \boldsymbol{\Omega}_k^{i+1} & 0 \\ \vdots & \vdots \\ \boldsymbol{\Omega}_{k-n+1}^0 \otimes \boldsymbol{\Omega}_k^{n-1} & 0 \\ 0 & -I \otimes \boldsymbol{H}_k^n \end{bmatrix} \quad (18)$$

which includes known matrices. The symbol ⊗ denotes the Kronecker product (element-by element multiplication of matrices) and in matrix $\boldsymbol{\theta}_k$, the below property is used as:

$$\boldsymbol{ABC} = (\boldsymbol{C}^T \otimes \boldsymbol{A})\boldsymbol{B} \quad (19)$$

$\boldsymbol{d}_k$ is also written as:

$$\boldsymbol{d}_k = \left[(\boldsymbol{C}^n_{k,k'}\delta(\Delta))^T \quad (\boldsymbol{C}^n_{k,k'}\delta(\Delta-1))^T \quad ,\ldots, \quad (\boldsymbol{C}^n_{k,k'}\delta(\Delta-\mathrm{n}))^T\right]^T \tag{20}$$

Notice that the vector $\boldsymbol{d}_k$ includes the unknown parameters of $\boldsymbol{Q}$ and $\boldsymbol{R}$. Thus, the solution method of LS cannot be considered unless accurate estimation of $\boldsymbol{d}_k$ exists. One of the efficient estimation methods is using the average of measurement in some time intervals for calculating the covariance in $\boldsymbol{d}_k$. In this approach, the current-time estimation is applied and then averaging on the time intervals is done while we have:

$$\boldsymbol{C}^n_{k,k'} = E\left(\tilde{\boldsymbol{e}}^n_k\left(\tilde{\boldsymbol{e}}^n_{k-k'}\right)^T\right) \approx \tilde{\boldsymbol{e}}^n_k\left(\tilde{\boldsymbol{e}}^n_{k-k'}\right)^T \tag{21}$$

and then, the averaging on all samples from *n+1* to *N*, we have:

$$\overline{\boldsymbol{C}}^n_{k,k'} = \frac{1}{N-n}\sum_{k=n+1}^{N}\boldsymbol{C}^n_{k,k'} \tag{22}$$

Thus, $\boldsymbol{d}_k$ is known and the equation (16) can be solved to obtain the accurate estimation of $\boldsymbol{W}_k$ including $\boldsymbol{Q}$ and $\boldsymbol{R}$ as:

$$\begin{gathered}\boldsymbol{\theta}_k\boldsymbol{W}_k = \boldsymbol{d}_k \\ \boldsymbol{W}_k = ({\boldsymbol{\theta}_k}^T\boldsymbol{\theta}_k)^{-1}(\boldsymbol{\theta}_k)^T\boldsymbol{d}_k\end{gathered} \tag{23}$$

The proposed algorithm depends on the n-sample prediction and measurement where increasing this sampling time causes more accurate estimation while increasing the complexity, simultaneously. Moreover, due to (22), it can be written as:

$$\overline{\boldsymbol{C}}^n_k = \frac{1}{N-n}\sum_{k=n+1}^{N}\boldsymbol{C}^n_{k,k'} = \overline{\boldsymbol{C}}^n_{k-1} + \boldsymbol{C}^n_k \tag{24}$$

Thus, the optimal estimation can be described in a recursive relation as below:

$$\boldsymbol{W}^{opt}_k = \boldsymbol{W}^{opt}_{k-1} + (\overline{\boldsymbol{\theta}}_k)^*(\overline{\boldsymbol{d}}_k - \boldsymbol{\theta}_k\boldsymbol{W}^*_{k-1}) \tag{25}$$

where $\boldsymbol{W}^{opt}$ represents the optimal vector of covariance matrices. The pseudocode of the algorithm is proposed in the below.

Pseudocode:

Step 1: define error in (4) to (9)

Step 2: calculate the expectation and covariance in (11) and (12)

Step 3: Using LS, covariance matrix estimation is done in (16), (17) and (23)

Step 4: Update the estimation process as in (25)

3.2 Method 2: Expectation and Estimation based on Kalman Filter

In this method, Kalman filter is reviewed and estimation of noise covariance matrix is done. We know that from Kalman filter:

$$\boldsymbol{e}_k = \boldsymbol{z}_k - \hat{\boldsymbol{z}}_k \tag{26}$$

and

$$\hat{\boldsymbol{x}}_{k+1|k+1} = \hat{\boldsymbol{x}}_{k+1|k} + \boldsymbol{K}_{k+1}\boldsymbol{e}_{k+1} \tag{27}$$

and if the convergence of estimation is guaranteed, we know from Eq. (1) and Eq. (26) that:

$$E\left(\left(\hat{\boldsymbol{x}}_{k+1|k+1} - \hat{\boldsymbol{x}}_{k+1|k}\right)\left(\hat{\boldsymbol{x}}_{k+1|k+1} - \hat{\boldsymbol{x}}_{k+1|k}\right)^T\right) = \boldsymbol{Q}_{k+1} \tag{28}$$

Due to (27), we can also write:

$$E\left(\left(\hat{\boldsymbol{x}}_{k+1|k+1} - \hat{\boldsymbol{x}}_{k+1|k}\right)\left(\hat{\boldsymbol{x}}_{k+1|k+1} - \hat{\boldsymbol{x}}_{k+1|k}\right)^T\right) = \boldsymbol{K}_{k+1}\boldsymbol{e}_{k+1}{\boldsymbol{e}_{k+1}}^T{\boldsymbol{K}_{k+1}}^T \tag{29}$$

So, one accurate estimation of $\boldsymbol{Q}$ can be stated considering (28) and (29) as:

$$\boldsymbol{Q}_{k+1} = \boldsymbol{K}_{k+1}\boldsymbol{e}_{k+1}{\boldsymbol{e}_{k+1}}^T{\boldsymbol{K}_{k+1}}^T \tag{30}$$

In similar way for covariance matrix of measurement noise ($\boldsymbol{R}$), we have:

$$\hat{\boldsymbol{z}}_{k+1|k+1} = \hat{\boldsymbol{z}}_{k+1|k} + \boldsymbol{e}_{k+1} \tag{31}$$

and if the convergence of estimation is guaranteed, we know that:

$$E\left(\left(\hat{\boldsymbol{z}}_{k+1|k+1} - \hat{\boldsymbol{z}}_{k+1|k}\right)\left(\hat{\boldsymbol{z}}_{k+1|k+1} - \hat{\boldsymbol{z}}_{k+1|k}\right)^T\right) = \boldsymbol{R}_{k+1} \tag{32}$$

and thus, the estimation of $\boldsymbol{R}$ can be described as:

$$\boldsymbol{R}_{k+1} = \boldsymbol{e}_{k+1}{\boldsymbol{e}_{k+1}}^T \tag{33}$$

Indeed due to (30) and (33), we have:

$$\boldsymbol{Q}_{k+1} = \boldsymbol{K}_{k+1}\boldsymbol{R}_{k+1}{\boldsymbol{K}_{k+1}}^T \tag{34}$$

The pseudocode of the algorithm is proposed in the below.

Pseudocode:

Step 1: define error in (26)

Step 2: applying Kalman filter equation in (27)

Step 3: define covariance matrix as in (30)

Step 4: calculate and update the covariance matrix as in (33) and (34).

3.3 Method 3: Gamma Distribution-based Estimation

A random variable (r.v.) $\mathbf{X}$, following Gamma distribution, with parameters $\boldsymbol{\Psi}$ and ν, is denoted by $\mathbf{X} \sim$ Gamma ($\boldsymbol{\Psi}$, ν) which can be defined by a probability density function (pdf) given by [24]:

$$f(\boldsymbol{X}) = \frac{\det(\boldsymbol{\Psi})^{\nu/2}}{2^{\frac{\nu p}{2}}\Gamma_p(v/2)}\det(\boldsymbol{X})^{-\frac{\nu+p+1}{2}}e^{-\frac{trace(\boldsymbol{\Psi X}^{-1})}{2}} \quad (35)$$

in which $\boldsymbol{\Psi} \in R^{p\times p}$ is a positive-definite scale matrix and ν is the number of degrees of freedom. $\Gamma_p(.)$ is the gamma function. The mean of this distribution is as [24]:

$$E(\boldsymbol{X}) = \frac{\boldsymbol{\Psi}}{\nu - p - 1} \quad (36)$$

The reason of selection this distribution is its strength in dynamic tracking. In addition, based on Gamma distribution, we can simultaneously estimate the unknown number, measurement rate states, kinematics states, extension states and tracks of extended targets in the presence of clutter, missed detection and data association uncertainty [27]. Moreover, effective management of inventory for fast-moving finished goods, even with disruptive lead times under both conventional and non-classical conditions can be employed [27]. Now due to the state-space model in (1) and (2), it can be describe as:

$$\hat{\boldsymbol{z}}_{k+1} = \boldsymbol{H}\hat{\boldsymbol{x}}_{k+1} \quad (37)$$

and we know that the distribution of $\hat{\boldsymbol{z}}_{k+1}$ is normal with mean of $\boldsymbol{H}\hat{\boldsymbol{x}}_{k+1}$ and the covariance matrix initially set by $\boldsymbol{S}_1$. Thus for $\hat{\boldsymbol{x}}_{k+1}$, we have a similar normal distribution with the initial variance as $\boldsymbol{P}_1$:

$$\hat{\boldsymbol{x}}_{k+1} \sim N(\boldsymbol{F}\hat{\boldsymbol{x}}_k, \boldsymbol{P}_1) \quad (38)$$

$$\boldsymbol{P}_1 = E((\boldsymbol{x}_k - \hat{\boldsymbol{x}}_k)(\boldsymbol{x}_k - \hat{\boldsymbol{x}}_k)^T)$$

Due to Bayes theorem, it can be written as:

$$f(\boldsymbol{P}_1|\hat{\boldsymbol{x}}_{k+1}) \propto \prod_{i=1}^{n_k} f(\hat{\boldsymbol{x}}_{k+1}|\boldsymbol{P}_1)f(\boldsymbol{P}_1)$$
$$= \det(\boldsymbol{P_1})^{-\frac{n_k}{2}}e^{-\frac{\sum_{i=1}^{n_k}\boldsymbol{e}_k^T\boldsymbol{K}_k^T\boldsymbol{P}_1^{-1}\boldsymbol{K}_k\boldsymbol{e}_k}{2}} \times \det(\boldsymbol{P_1})^{-\frac{v+p-1}{2}}e^{-\frac{trace(\boldsymbol{P}_1^{-1})}{2}} \quad (39)$$

and thus, we have:

$$f(\boldsymbol{P}_1|\hat{\boldsymbol{x}}_{k+1}) = \det(\boldsymbol{P}_1)^{-\frac{n_k+p+1+v}{2}}e^{-\frac{trace((\boldsymbol{P}^*+\boldsymbol{\Psi_0})\boldsymbol{P}_1^{-1})}{2}} \quad (40)$$

where we have:

$$\boldsymbol{P}^* = \sum_{k=1}^{n_k}(\hat{\boldsymbol{x}}_{k+1} - \boldsymbol{F}\hat{\boldsymbol{x}}_k)(\hat{\boldsymbol{x}}_{k+1} - \boldsymbol{F}\hat{\boldsymbol{x}}_k)^T \quad (41)$$

and thus, $\boldsymbol{P}^*$ can be written as:

$$\boldsymbol{P}^* = \boldsymbol{F}\boldsymbol{P}_{\boldsymbol{k}|\boldsymbol{k}}\boldsymbol{F}^{\boldsymbol{T}} + \boldsymbol{Q}_{\boldsymbol{k}} \quad (42)$$

Finally, the estimation of $\boldsymbol{Q}$ can be presented as below:

$$\boldsymbol{Q}_{\boldsymbol{k}} = \boldsymbol{P}^* - \boldsymbol{F}\boldsymbol{P}_{\boldsymbol{k}|\boldsymbol{k}}\boldsymbol{F}^{\boldsymbol{T}} \quad (43)$$

The similar treatment can be done for Matrix $\boldsymbol{R}$ estimation and finally, we have:

$$\boldsymbol{R}_{\boldsymbol{k}} = \boldsymbol{S}^* - \boldsymbol{H}\boldsymbol{P}_{\boldsymbol{k}|\boldsymbol{k}}\boldsymbol{H}^{\boldsymbol{T}} \quad (44)$$

where

$$\boldsymbol{S}^* = \sum_{k=1}^{n_k}(\boldsymbol{z}_{k+1} - \boldsymbol{H}\hat{\boldsymbol{x}}_{k+1|k+1})(\boldsymbol{z}_{k+1} - \boldsymbol{H}\hat{\boldsymbol{x}}_{k+1|k+1})^T \quad (45)$$

The pseudocode of the algorithm is proposed in the below.

Pseudocode:

Step 1: define covariance due to Gamma distribution in (38)

Step 2: define the function of Gamma distribution based on Covariance in (40) to (42)

Step 3: calculate the covariance matrix as in (43) and (44)

4 Discussion on three proposed methods

Three methods are proposed in previous section for noise covariance matrix estimation in state and measurement model. In the first method (n-step prediction), the efficient least square is used finally to obtain the accurate estimation of covariance matrix. However, the inverse of matrix and n-step prediction have complexities in practical using of this approach. Recursive relation to find the accurate estimation is also mentioned in this method. In the second method (Kalman gain covariance), the assumption of convergence must be guaranteed which is the main challenge of this strategy. Considering the convergence condition, this second approach provides an efficient estimation. Discussing on the third method (Gamma-distribution noise estimation), the suitable distribution is considered for covariance matrix which leads to an accurate estimation. This method needs no inverse matrix and has less complexity compared to the first method. Also due to the probability density function (pdf), the convergence is ensured which results in an acceptable estimation performance. Table 1 summarizes the properties of these methods which provides an efficient comparison among them. In the next section, simulation is done to provide a fair comparison among three proposed methods and other efficient approaches in this regard.

Table 1. The comparison table of three proposed methods and their characteristics

Property / Method	Complexity	Accuracy	Convergence
n-step Prediction	High	Medium/High	Ensured
Kalman Gain	High	High	Assumed by default
Gamma Distribution	Low	High	Ensured

5 Simulation and Comparison

In this section, the simulation is done based on the performance comparison among three proposed methods and presented method in [23] and linear Kalman filter. These comparisons are based on the target tracking performance including two different trajectories in two scenarios. The time is set to $T = 4\ s$ where the samples are adjusted in this time duration with sampling frequency of 45 KHz. In this regard, the system model is described as:

$$F = \begin{bmatrix} 1 & T & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & T \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{46}$$

$$x = \begin{bmatrix} p_x \\ v_x \\ p_y \\ v_y \end{bmatrix}$$

The state vector includes the position and velocity in x-y space. For the simulation, 40 Monte Carlo runs are done where the main considerations in the linear Kalman filter are stated as:

$$\begin{aligned} &\boldsymbol{P}_{0|0} = 1500^2 I \\ &\boldsymbol{Q}_k = \boldsymbol{Q} = 100^2 I \\ &\boldsymbol{R}_k = \boldsymbol{R} = 1000^2 I \\ &\hat{x}_{0|0} = [-4e4 \quad 100\ m/s \quad 5e4 \\ &\qquad\qquad - 120\ m/s]^T \end{aligned} \tag{47}$$

where the initial velocities in x-y direction are also set in this simulation. Two scenarios including two different trajectories are tested to view the performance of the proposed methods in target tracking. In the first scenario as described in Figure 1, the direct line is the selected trajectory with the characteristic of:

Start Point = [-4e4,5e4]

End Point = [4e4,-3e4]

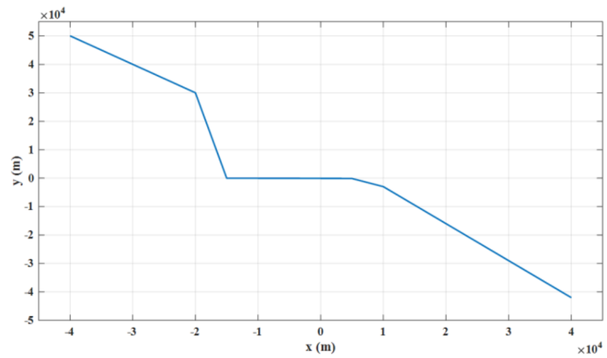

Fig 1. The direct line trajectory in Scenario 1

In Figure 2, the result of tracking of proposed methods in comparison with each other and the method in [23] and linear Kalman filter is depicted. As shown in this figure, the Gamma-distribution noise estimation (proposed method 3) has better estimation and can better track the selected trajectory. Other proposed methods also efficient compared to the linear Kalman filter and method in [23]. It is worth mentioning that the proposed methods efficiently track the direct-line trajectory. In addition, the RMSE of the estimation of the presented methods is described in Table 2. As can be seen from this table, the RMSE of method 3 is the least one compared to other methods. Figure 3 shows the RMSE of the mentioned methods versus time. The result of this figure verifies the result of Table 2. Moreover, the noise covariance values versus time ($\boldsymbol{Q}$) is plotted in Figure 4. As depicted in this figure, the proposed methods are converged to the actual value in progressing time process of tracking.

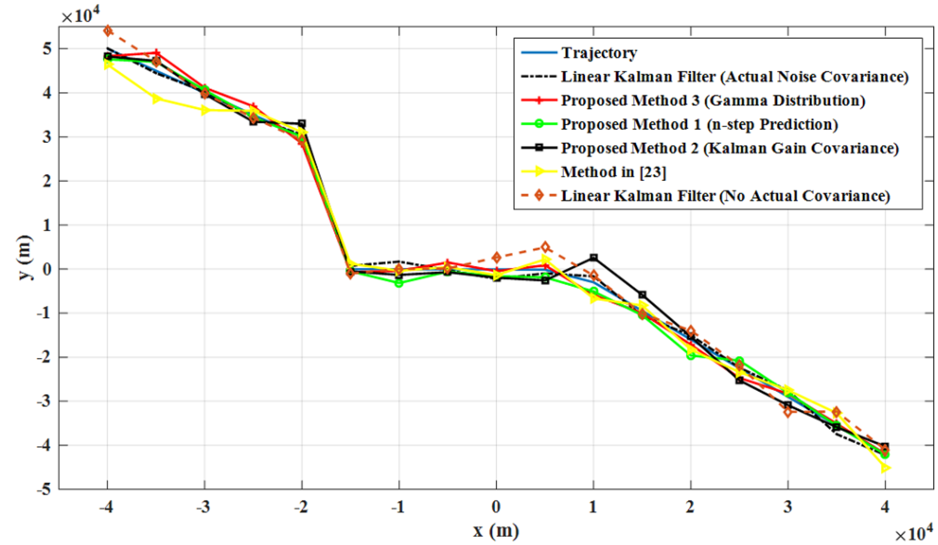

Fig 2. Tracking the direct line applying the proposed methods and comparison with other methods

Table 2. Comparison of RMSE of the discussed methods

Method	**RMSE** ($\sqrt{d_x^2 + d_y^2}$)
Linear Kalman Filter	1204.48
Method in [23]	1113.75
Proposed method 2	1042.45
Proposed method 1	1006.76
Proposed method 3	989.28
Linear Kalman Filter (Actual Covariance)	967.35

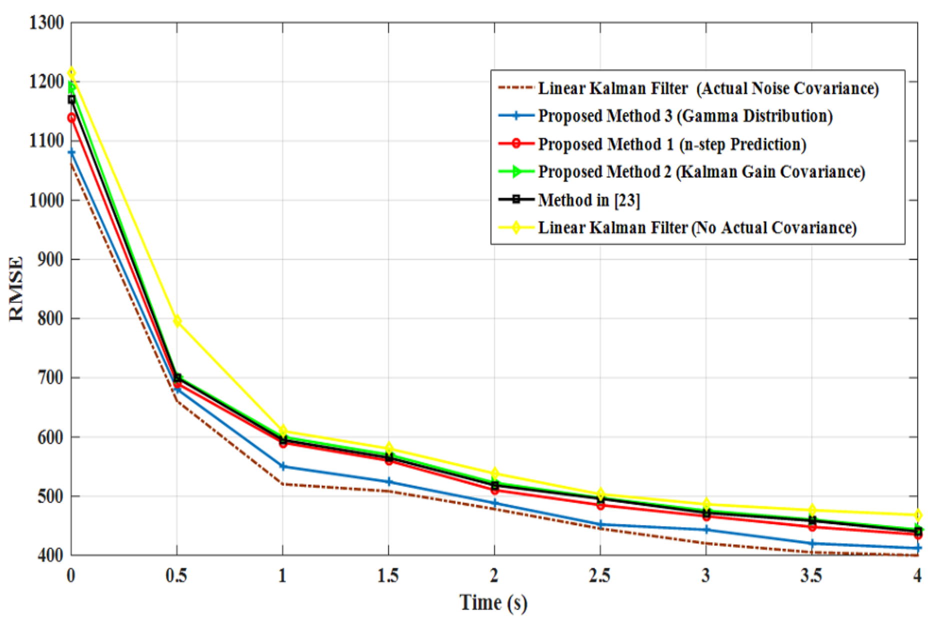

Fig 3. RMSE of the proposed methods in time and comparison with other methods

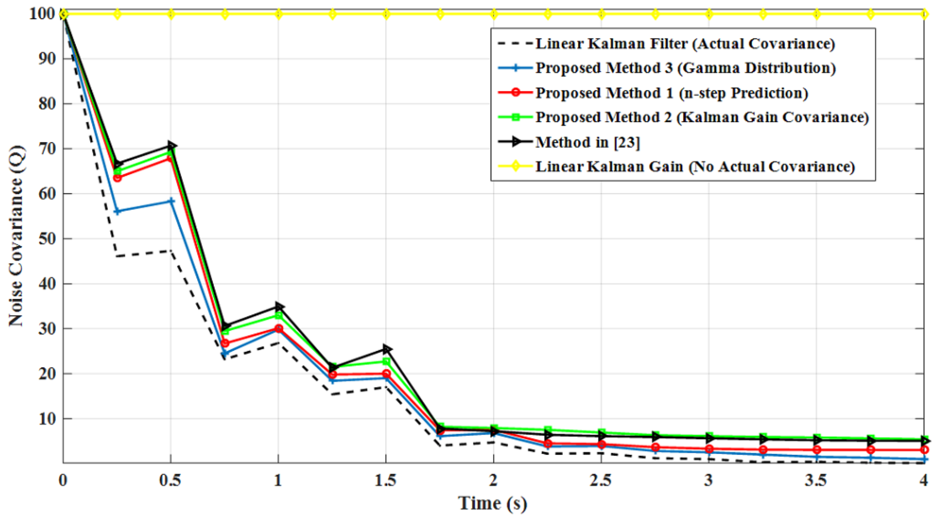

Fig 4. Noise covariance of the proposed methods in time and comparison with other methods

In the second scenario, the trajectory is a combination of several direct lines which can generate approximately a curve as described in Figure 5. The start and end points are as below:

Start: [-6e4, -17.6e4]

End: [0, 0]

However, the approach of the proposed methods is based on the tracking of the combination of direct lines where the complexity is increased. Because of the complexity of the assumed trajectory, the performance of the proposed methods is notable in this regard as shown in Figure 6.

Gamma-distribution noise estimation is more efficient than other methods in accurate tracking with less RMSE. RMSE is also described in Table 3 for this scenario. Obviously, the error is increased due to the complexity of trajectory and estimation. It is worth mentioning that the RMSE of the proposed method 3 is less than other methods as can be understood from the table.

In addition, the sample efficient work of the Gamma-distribution noise estimation in tracking the target is depicted in Figure 7. The tracking path is a complex one where the performance of the third method (method 3) is notable and efficient. Thus, the proposed method can reliably track the complex tracks.

As the final comparison, the efficient work of the second and third proposed methods is compared with the linear Kalman filter in a simple track in Figure 8. As depicted in this figure, the Gamma-distribution noise estimation accurately tracks the path and follow the performance of ideal kalman filter method.

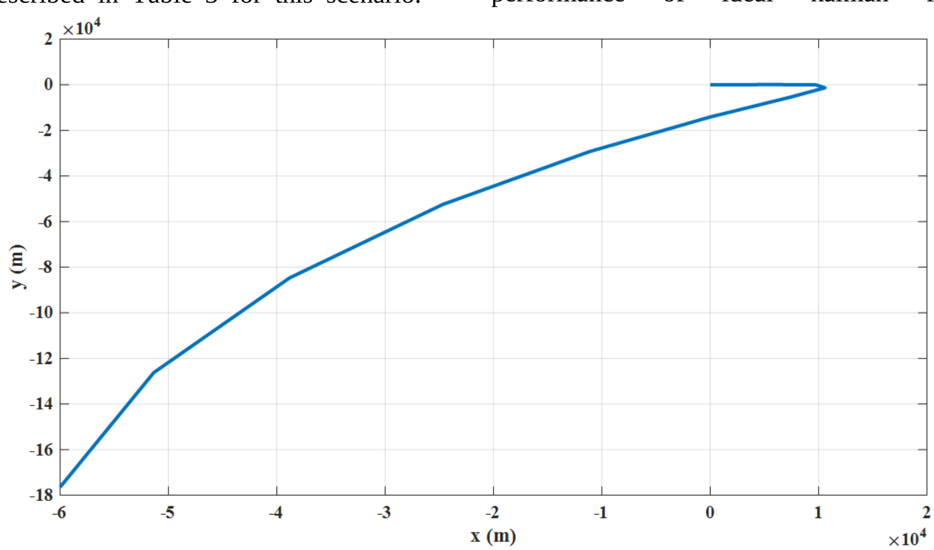

Fig 5. The combination of several direct lines in trajectory of Scenario 2.

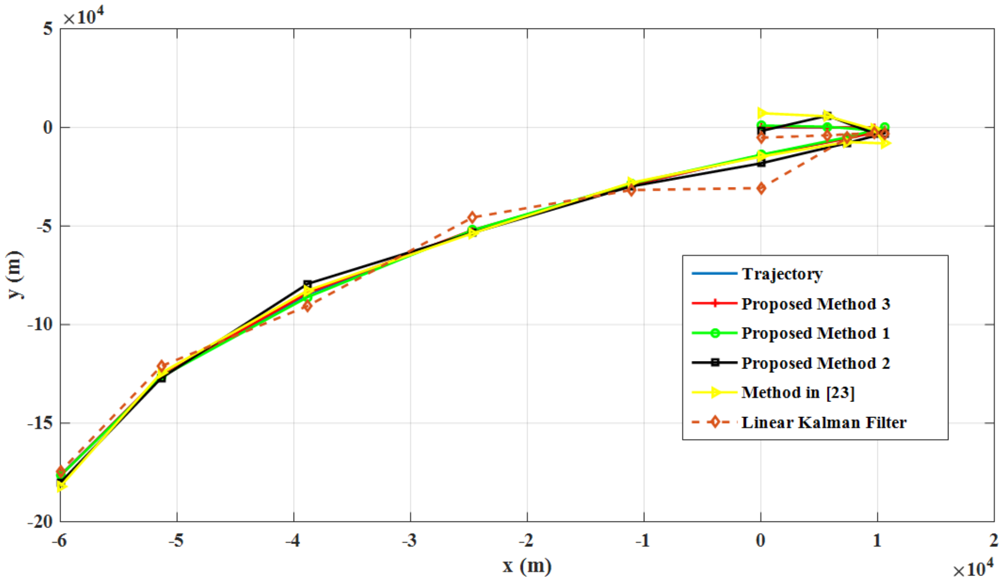

Fig 6. Tracking the semi-curve-based trajectory with applying the proposed methods and comparison with other methods.

Table 3. Comparison of RMSE of the discussed methods

Method	RMSE ($\sqrt{d_x^2 + d_y^2}$)
Linear Kalman Filter	154.13
Method in [23]	141.87
Proposed method 2	146.45
Proposed method 1	132.57
Proposed method 3	124.65

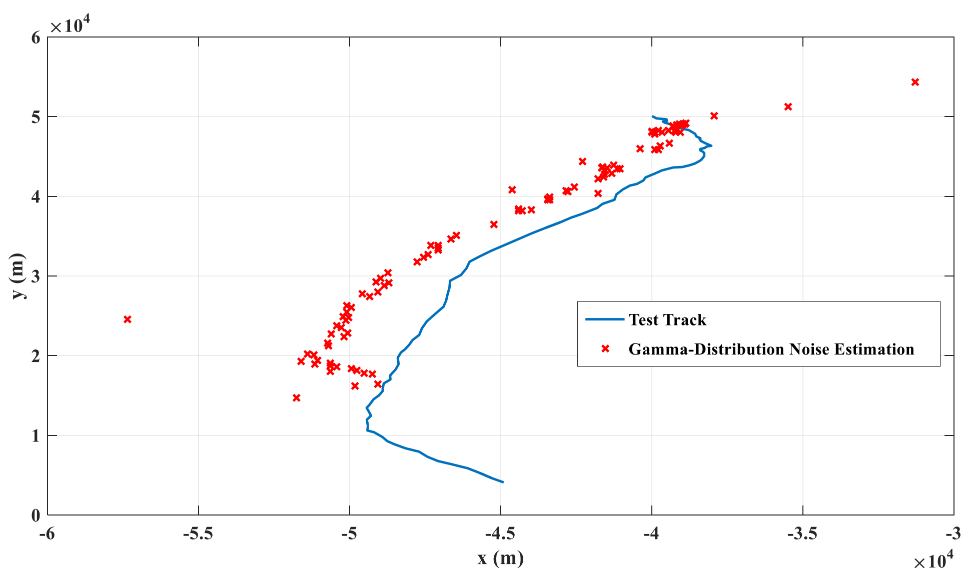

Fig 7. Sample work of efficient tracking by the proposed strategy for a complex track test (method 3)

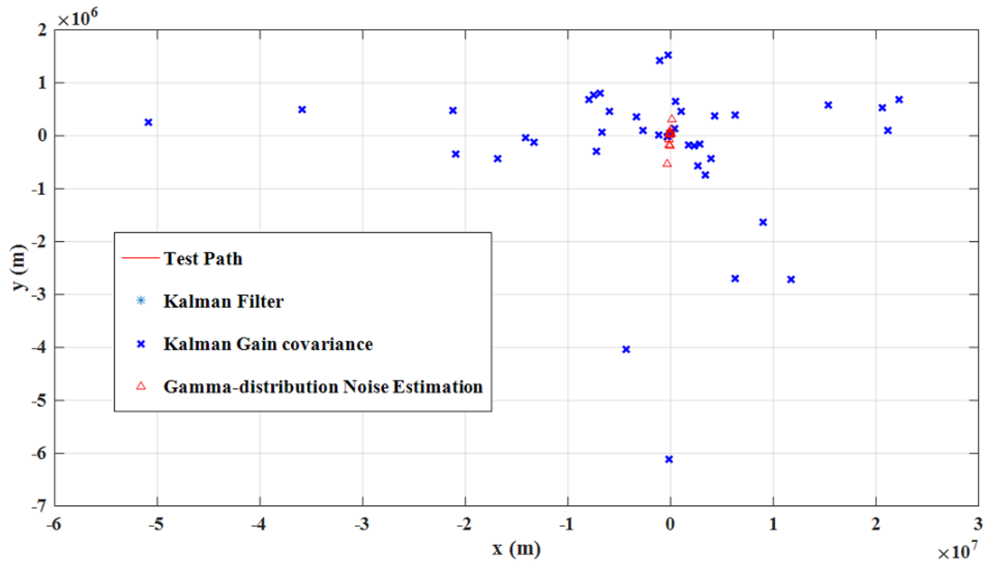

Fig 8. Sample work of efficient tracking by the proposed strategy for a simple track test (method 3)

6 Conclusion

In this paper, three proposed methods are applied to achieve the accurate noise covariance matrices in target tracking projects. Method 1 results in an accurate recursive algorithm with acceptable simplicity. Method 2 is based on Kalman gain covariance where it is very simple compared to other methods. Method 3 is based on Gamma distribution which is low-complex and most efficient as shown in simulation results. Compared with other methods in the literature and linear Kalman filter, all three methods have notable performance and low complexity where RMSE is low in these proposed methods. The RMSE in the Gamma-distribution-based method is 124.65 which is the least value among other proposed methods and works of the compared literature.

Conflict of Interest Statement

There is no conflict of interest in submitting the paper to the journal.

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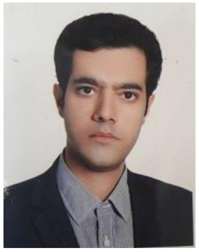

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