Singular Perturbation Theory in Output Feedback Control of Pure-Feedback Systems

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Abstract: this paper studies output feedback control of pure-feedback systems with immeasurable states and completely non-affine property. Since availability of all the states is usually impossible in the actual process, we assume that just the system output is measurable and the system states are not available. First, to estimate the immeasurable states a state observer is designed. Relatively fewer results have been proposed for pure-feedback systems because the cascade and non-affine properties of pure-feedback systems make it difficult to find the explicit virtual controls and actual control. Therefore, by employing the singular perturbation theory in back-stepping control procedure, the virtual/actual control inputs are derived from the solutions of a series of fast dynamical equations which can avoid the "explosion of complexity'' inherently existing in the conventional back-stepping design. The stability of the resulting closed-loop system is proved by Tikhonov's theorem in the singular perturbation theory. Finally, the detailed simulation results are provided to demonstrate the effectiveness of the proposed controller, which can overcome the non-affine property of pure-feedback systems with lower complexity and fewer design parameters.

Keywords: Nonlinear Pure-Feedback Systems, Output Feedback Control, State Observer, Singular Perturbation Theory

1 Introduction

The pure-feedback system represents a more general class of triangular systems, which have no affine appearance of the variables to be used as virtual controls. In practice, there are many systems falling into this category, such as mechanical systems, aircraft flight control systems, biochemical process, Duffing oscillator, etc. For a pure-feedback system, the backstepping control technique [1] provides a systematic framework for the design of tracking and regulation strategies with its constructive Lyapunov design procedures. However, the difficulties associated with the backstepping controller design for the pure-feedback system lie in that, 1) no dynamical inverted virtual/actual control inputs can be found explicitly and, 2) the problem of "explosion of complexity" inherently exists in the conventional back-stepping design because of the repeated differentiations of the virtual control inputs.

Adaptive neural networks (NNs) control schemes in [2] and [3] are first proposed for a class of pure-feedback nonlinear systems, where the last equation of the controlled system is an affine nonlinear to avoid the algebraic loop problem. In [4-7] control of the completely non-affine pure-feedback systems are investigated. These papers employ the function approximation technique using NNs or fuzzy logic systems (FLs) to compensate unknown nonlinear terms in the control system design. In the previous studies [2-7], the authors use the NNs or FLs to approximate the ideal virtual/actual control inputs. However, the time derivatives of the virtual control inputs are either ignored completely, leading to a poor tracking performance or approximated by the NNs or FLs, resulting in a complicated controller design. Moreover, these methods are suitable to the case in which the plant models are completely unknown and they cannot take advantage of the prior knowledge of the system even the plant model is exactly known. To overcome the "explosion of complexity" problem, dynamic surface control (DSC) was proposed in the controller design for non-affine system, by employing first-order filtering of the synthetic virtual control input at each step of backstepping approach [8]. However, this method will produce an algebraic loop, because the controller is employed in the approximation algorithm, whose output is simultaneously utilized in the controller. The problem of algebraic loop is solved in [9] and [10] by combining

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the DSC technique and input-to-state stability- (ISS-) modular design method [4]. However, in [9] and [10], the corresponding neural network (NN) and filter are needed to be designed in each recursive step. Thus, the method is still complex.

The combination of the backstepping and time scale separation for completely non-affine pure-feedback systems is proposed in [11, 12]. In this proposed technique, first by employing the time scale separation method, the time derivatives of the virtual/actual control inputs are defined as solutions of fast dynamic equations and then, their integrals are used as the virtual/actual control inputs. In this technique, the two mentioned problems of backstepping are solved simultaneously. However, the designed controller in [11, 12] is under the assumption that the full states are known and it is usually impossible that all the states are available in the actual process.

Despite these efforts in in control of pure feedback systems, the above mentioned methods are all based on the assumption that the states of the controlled systems are available for measurement. It is well known that state variables are often immeasurable for many practical nonlinear systems. However, few attempts have been made on control of pure-feedback systems with immeasurable states, which are important and more practical.

In [13-15] a new adaptive fuzzy output feedback control approach is developed in which, FLs are utilized to approximate the unknown nonlinear functions; and the filtered signals are introduced to circumvent algebraic loop systems encountered in the implementation of the controller, and a fuzzy state adaptive observer is designed to estimate the immeasurable states. Recently, a low-complexity and free backstepping like approach is proposed in [16] based on sliding mode control theory. In this technique, a novel transformation method is included, which can transform the state-feedback control of non-affine systems into output feedback control of strict-feedback affine systems. However, in [16], it is assumed that the bounds of the derivatives of the nonlinear functions for all the variables are known. Hence, for non-affine systems, it needs to further study the low-complexity controller design that needs less restrictive conditions and system knowledge.

In this paper, we present the output feedback control problem for completely non-affine pure-feedback systems with immeasurable states using the backstepping and time-scale separation technique. For the controller design, we first design the state observer to estimate the immeasurable states. Then, the time-scale separation concept and the backstepping technique are combined to develop the approximate version of virtual/actual control laws. Therefore, the virtual/actual controllers are defined as solutions of fast dynamic equations which accomplish time-scale separation between the state observer and controllers. It is proven that the proposed control approach can guarantee that the observer and tracking errors converge to a small neighborhood of the origin. Finally, the simulation results are provided to validate the effectiveness of the proposed approach.

2 Problem Formulation and Preliminaries

2.1 Problem Statement

Consider the following pure-feedback nonlinear system:

$$\dot{x}_i = f_i(\underline{x}_i, x_{i+1}) + x_{i+1} \qquad i = 1, 2, \dots, n-1$$
$$\dot{x}_n = f_n(\underline{x}_n, u) + u$$
$$y = x_1 \tag{1}$$

where $\underline{x}_i = [x_1, x_2, \dots, x_i]^T \in R^i, i = 1, 2, \dots, n$ is the system state vector, $u \in R$ and $y \in R$ are system input and output, respectively; $f_i(.), i = 1, 2, \dots, n$ are smooth nonlinear functions. In this paper, it is assumed that output y is available for measurement.

Assumption 1: There exist a set of constants m_i, $i = 1, \dots, n$, $\forall X_1, X_2 \in R^i$, such that the following inequality holds.

$$|f_i(X_1) - f_i(X_2)| \le m_i \| X_1 - X_2 \| \tag{2}$$

where$\| X_1 - X_2 \|$ expresses the two-norm of vector $X_1 - X_2$.

Assumption 2: $(\partial f_i/\partial x_{i+1})$ and$(\partial f_n/\partial \mathrm{u})$ are bounded away from zero for $\underline{x}_{i+1} \in \Omega_{\underline{x}_{i+1}} \subset D_{\underline{x}_{i+1}}$and $(\underline{x}_n, u) \in \Omega_{\underline{x}_n,u} \subset D_{\underline{x}_n} \times D_u$, where $\Omega_{\underline{x}_{i+1}}$and $\Omega_{\underline{x}_n,u}$ are compact sets; that is $(\partial f_i/\partial x_{i+1})$ and$(\partial f_n/\partial \mathrm{u})$are either positive or negative. Without losing the generality, we assume $(\partial f_i/\partial x_{i+1}) > 0$ and $(\partial f_n/\partial \mathrm{u}) > 0$.

Remark 1: Note that the constants m_i in Assumption 1 are only used for the purpose of the stability of the control system, instead of being used to implement the controller. Therefore, the constants m_i are only required to exist, they may be unknown. Moreover, they do not need to be estimated in implementing the controller [14,15].

In this paper, it is assumed that the states x_i, $i = 2, \dots, n$, are not available for the controller design. Our control objective is to design an output feedback control scheme by using time scale separation in backstepping procedure so that the state observer and tracking errors are as small as desired.

2.2 Preliminaries on Singular Perturbation Theory

For proving our main result, we will use Tikhonov's theorem on singular perturbations, which we recall below [17]. Consider the problem of solving the state equation

$$\dot{x}(t) = f(t, x(t), z(t), \varepsilon) \qquad x(0) = \xi(\varepsilon)$$
$$\varepsilon\dot{z}(t) = g(t, x(t), z(t), \varepsilon) \qquad z(0) = \eta(\varepsilon) \tag{3}$$

where ε is a "small" scalar parameter. $\xi(\varepsilon)$ and $\eta(\varepsilon)$ are smooth. It is assumed that the functions f and g are continuously differentiable in their arguments for $(t,x,z,\varepsilon) \in [0,\infty) \times D_x \times D_z \times [0,\varepsilon_0]$ where $D_x \subset R^n$ and $D_z \subset R^m$ are open connected sets, $\varepsilon_0 >> 0$. If $g(t,x,z,0) = 0$ has $l \geq 1$ isolated real roots $z = h_a(t,x)$, $a = 1,2,\dots,l$, for each $(t,x) \in [0,\infty) \times D_x$ when $\varepsilon = 0$, we say that the model (3) is in 'standard form'. Let us choose one fixed parameter $a \in \{1,\dots,l\}$, and drop the subscript a from h from now on. Let $v = z - h(t,x)$. From singular perturbation theory, the 'reduced system' is represented by

$$\dot{x}(t) = f(t, x(t), h(t, x(t)), 0) \qquad x(0) = \xi(0) \tag{4}$$

and the 'boundary layer system' with the new time scale $\tau = t/\varepsilon$ is defined as

$$\frac{dv}{d\tau} = g\big(t, x, v + h(t, x(t)), 0\big)$$

$$v(0) = \eta_0 - h(0, \xi_0) \tag{5}$$

where $\eta_0 = \eta(0)$ and $\xi_0 = \xi(0)$, $(t,x) \in [0,\infty) \times D_x$ are treated as fixed parameters.

3 State Observer Design

Note that the states $x_2, x_3, \dots, x_n$ in system (1) are not available for feedback; therefore, a state observer should be established to estimate the unmeasured states, and then output feedback control scheme is investigated based on the designed state observer. To begin with, rewrite (1) as

$$\dot{x}_i = f_i(\underline{\hat{x}}_i, \hat{x}_{i+1}) + x_{i+1} + \Delta f_i \qquad i = 1,2,\dots,n-1$$

$$\dot{x}_n = f_n(\underline{\hat{x}}_n, u) + u + \Delta f_n$$

$$y = x_1 \tag{6}$$

where $\Delta f_i = f_i(\underline{x}_i, x_{i+1}) - f_i(\underline{\hat{x}}_i, \hat{x}_{i+1})$, $i = 1,2,\dots,n-1$; $\Delta f_n = f_n(\underline{x}_n, u) - f_n(\underline{\hat{x}}_n, u)$; $\underline{\hat{x}}_i$ is the estimate of the state vectors $\underline{x}_i$, which can be obtained by the state observer designed later. Rewrite (6) in the state space form

$$\underline{\dot{x}}_n = A\underline{x}_n + Ky + \sum_{i=1}^{n-1} B_i\,[f_i(\underline{\hat{x}}_i, \hat{x}_{i+1}) + \Delta f_i] + B_n[f_n(\underline{\hat{x}}_n, u) + u + \Delta f_n]$$

$$= A\underline{x}_n + Ky + \textstyle\sum_{i=1}^{n} B_i\,[f_i(\underline{\hat{x}}_i, \hat{x}_{i+1}) + \Delta f_i] + B_n u \tag{7}$$

where $A = \begin{bmatrix} -k_1 & & \\ \vdots & I & \\ -k_n & 0 \;\dots & 0 \end{bmatrix}$, $K = \begin{bmatrix} k_1 \\ \vdots \\ k_n \end{bmatrix}$, and $B_n = \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}$, $B_i = [0 \underbrace{\dots\ 1}_{i} \dots\ 0]^T$.

Choose the vector K to make matrix A a strict Hurwitz matrix. Given a matrix $Q = Q^T > 0$, there exists a matrix $P = P^T > 0$satisfying

$$A^T P + PA = -2Q \tag{8}$$

Design a state observer as

$$\dot{\hat{x}}_i = \hat{x}_{i+1} + k_i(y - \hat{x}_1) + f_i(\hat{x}_i, \hat{x}_{i+1})$$

$$i = 1,2,\dots,n-1$$

$$\dot{\hat{x}}_n = k_n(y - \hat{x}_1) + f_n(\underline{\hat{x}}_n, u) + u \tag{9}$$

$$\hat{y} = \hat{x}_1$$

Rewrite (9) as

$$\underline{\dot{\hat{x}}}_n = A\underline{\hat{x}}_n + Ky + \sum_{i=1}^{n} B_i\, f_i(\underline{\hat{x}}_i, \hat{x}_{i+1}) + B_n u$$

$$\hat{y} = C\underline{\hat{x}}_n \tag{10}$$

where $C = [1 \;\; \dots \;\; 0 \;\; \dots \;\; 0]$.

Let $= \underline{x}_n - \underline{\hat{x}}_n$be observer error. Then from (7) and (10), one has

$$\dot{e} = Ae + \textstyle\sum_{i=1}^{n} B_i\,\Delta f_i = Ae + \Delta F \tag{11}$$

where $\Delta F = [\Delta f_1, \dots, \Delta f_n]^T$.

Consider the Lyapunov function candidate V_0 as

$$V_0 = \tfrac{1}{2} e^T P e \tag{12}$$

The time derivation of V_0 is

$$\dot{V}_0 = \tfrac{1}{2}\dot{e}^T P e + \tfrac{1}{2} e^T P \dot{e} \tag{13}$$

Using (8) and substituting (11) into (13) results in

$$\dot{V}_0 \leq -\lambda_{min}(Q)\|e\|^2 + e^T P \Delta F \tag{14}$$

By the Young's inequality $2ab \leq a^2 + b^2$, and by Assumption 1, one can obtain the following inequalities:

$$|e^T P \Delta F| \leq \tfrac{1}{2}\|e\|^2 + \tfrac{1}{2}\|P\|^2\|\Delta F\|^2 \leq \tfrac{1}{2}\|e\|^2 + \tfrac{1}{2}\|P\|^2(|\Delta f_1|^2 + \dots + |\Delta f_n|^2)$$

$$\leq \tfrac{1}{2}\|e\|^2 + \|P\|^2 \textstyle\sum_{i=1}^{n} {m_i}^2\, \|e\|^2 \tag{15}$$

Substituting (15) into (14) yields

$$\dot{V}_0 \leq -r_1\|e\|^2 \tag{16}$$

where $r_1 = \lambda_{min}(Q) - 1 - \|P\|^2 \sum_{i=1}^{n} {m_i}^2$.

Remark 2. The designed state observer (9) can guarantee the convergences of the observer errors if we choose $r_1 > 0$. Thus the designed state observer of this paper is reasonable [14, 15].

4 Output Feedback Control Design and Stability Analysis

In this section, an output feedback controller will be developed by using the above state observer in the framework of the combination of the backstepping technique and singular perturbation theory. Similar to the backstepping method, this design procedure contains n steps. Employing time-scale separation concept, the virtual control laws α_i, $i = 1, \dots, n-1$ and finally, the actual control law u are obtained.

The design procedure is presented in the following. Define the error variables as $z_1 = y - y_r$ and $z_i = \hat{x}_i - \alpha_{i-1}$ where $i = 2, \ldots, n$.
Step 1: Expressing x_2 in terms of its estimate as $x_2 = \hat{x}_2 + e_2$, we obtain

$$\dot{z}_1 = \dot{x}_1 - \dot{y}_r = x_2 + f_1(x_1, x_2) - \dot{y}_r = \hat{x}_2 + f_1(\hat{x}_1, \hat{x}_2) - \dot{y}_r + e_2 + \Delta f_1 \quad (17)$$

Taking $\hat{x}_2$ as a virtual control, one has

$$\dot{z}_1 = z_2 + \alpha_1 + f_1(\hat{x}_1, z_2 + \alpha_1) - \dot{y}_r + e_2 + \Delta f_1 \quad (18)$$

Then, α_1 as the first virtual controller can be specified as the solution of

$$\alpha_1 + f_1(\hat{x}_1, z_2 + \alpha_1) - \dot{y}_r = -c_1 z_1 \quad (19)$$

where $c_1 > 1$ is the first control gain. However, in (19), because of the non-affine property of the non-linear functions, α_1 cannot be explicitly computed. According to the following fast dynamics based on time-scale separation concept, an approximate virtual controller is designed

$$\varepsilon_1 \dot{\alpha}_1 = -\text{sign}\left(\frac{\partial Q_1}{\partial \alpha_1}\right) Q_1(t, \bar{z}_2, \alpha_1) \quad (20)$$

along with the initial condition $\alpha_1(0) = \alpha_{1,0}$, $\varepsilon_1 \ll 1$, $\bar{z}_2 = [z_1, z_2]^T$,

$$Q_1(t, \bar{z}_2, \alpha_1) = c_1 z_1 + \alpha_1 + f_1(\hat{x}_1, z_2 + \alpha_1) - \dot{y}_r \quad (21)$$

If $\alpha_1 = h_1(t, \bar{z}_2)$ be an isolated root of $Q_1(t, \bar{z}_2, \alpha_1) = 0$, then the reduced system is defined as

$$\dot{z}_1 = -c_1 z_1 + z_2 + e_2 + \Delta f_1 \qquad z_1(0) = z_{1,0} \quad (22)$$

and the boundary layer system can be represented by

$$\frac{dv_1}{d\tau_1} = -\text{sign}\left(\frac{\partial Q_1}{\partial \alpha_1}\right) Q_1(t, \bar{z}_2, v_1 + h_1(t, \bar{z}_2)) \quad (23)$$

where $v_1 = \alpha_1 - h_1(t, \bar{z}_2)$ and $\tau_1 = t/\varepsilon_1$.

Considering the control Lyapunov function $V_1 = V_0 + \frac{1}{2} z_1^2$ and using the reduced system (22), it is deduced that

$$\begin{aligned}\dot{V}_1 &= \dot{V}_0 + z_1[-c_1 z_1 + z_2 + e_2 + \Delta f_1] \\ &\leq -r_1 \|e\|^2 - c_1 z_1^2 + z_1 z_2 + z_1 e_2 + |z_1 \Delta f_1| \end{aligned} \quad (24)$$

By Young's inequality and Assumption 1, one has

$$z_1 e_2 \leq \frac{1}{2} z_1^2 + \frac{1}{2} |e_2|^2 \leq \frac{1}{2} z_1^2 + \frac{1}{2} \|e\|^2 \quad (25)$$

$$|z_1 \Delta f_1| \leq \frac{1}{2} z_1^2 + \frac{1}{2} |\Delta f_1|^2 \leq \frac{1}{2} z_1^2 + {m_1}^2 \|e\|^2 \quad (26)$$

Substituting (25) and (26) into (24) yields

$$\dot{V}_1 \leq -r \|e\|^2 - (c_1 - 1) z_1^2 + z_1 z_2 \quad (27)$$

where $r = r_1 - \frac{1}{2} - {m_1}^2$.
Steps i ($i = 2, \ldots, n-1$): The time derivative of z_i is expressed as

$$\begin{aligned}\dot{z}_i &= \dot{\hat{x}}_i - \dot{\alpha}_{i-1} = \hat{x}_{i+1} + k_i e_1 + f_i(\underline{\hat{x}}_i, \hat{x}_{i+1}) - \dot{\alpha}_{i-1} \\ &= z_{i+1} + \alpha_i + k_i e_1 + f_i(\underline{\hat{x}}_i, z_{i+1} + \alpha_i) - \dot{\alpha}_{i-1} \end{aligned} \quad (28)$$

Similar to step 1, we should find α_i such that

$$\alpha_i + k_i e_1 + f_i(\underline{\hat{x}}_i, z_{i+1} + \alpha_i) - \dot{\alpha}_{i-1} = -c_i z_i - z_{i-1} \quad (29)$$

where $c_i > 0$ is the ith positive control gain. To overcome the non-affine property, the ith approximate virtual controller can be designed as the following ith fast dynamics

$$\varepsilon_i \dot{\alpha}_i = -\text{sign}\left(\frac{\partial Q_i}{\partial \alpha_i}\right) Q_i(t, \bar{z}_{i+1}, \alpha_i) \quad (30)$$

where $\alpha_i(0) = \alpha_{i,0}$, $\varepsilon_i \ll 1$, $\bar{z}_{i+1} = [z_1, \ldots, z_{i+1}]^T$, and

$$\begin{aligned}Q_i(t, \bar{z}_{i+1}, \alpha_i) &= c_i z_i + z_{i-1} + \alpha_i + k_i e_1 \\ &+ f_i(\underline{\hat{x}}_i, z_{i+1} + \alpha_i) - \dot{\alpha}_{i-1} \end{aligned} \quad (31)$$

Remark 3: In this step, the time derivative of the virtual control input $\dot{\alpha}_{i-1}$ is appeared which has been designed in the previous step $\dot{\alpha}_{i-1} = -\text{sign}\left(\frac{\partial Q_{i-1}}{\partial \alpha_{i-1}}\right) Q_{i-1}(t, \bar{z}_i, \alpha_{i-1})/\varepsilon_{i-1}$. Therefore, the "explosion of complexity" arising from the calculation of this term is avoided [12].

Let $\alpha_i = h_i(t, \bar{z}_{i+1})$ be an isolated root of $Q_i(t, \bar{z}_{i+1}, \alpha_i) = 0$. Then the reduced system is defined as

$$\dot{z}_i = -c_i z_i - z_{i-1} + z_{i+1} \qquad z_i(0) = z_{i,0} \quad (32)$$

and the boundary layer system can be represented by

$$\frac{dv_i}{d\tau_i} = -\text{sign}\left(\frac{\partial Q_i}{\partial \alpha_i}\right) Q_i(t, \bar{z}_{i+1}, v_i + h_i(t, \bar{z}_{i+1})) \quad (33)$$

where $v_i = \alpha_i - h_i(t, \bar{z}_{i+1})$ and $\tau_i = t/\varepsilon_i$. Considering the control Lyapunov function $V_i = V_{i-1} + \frac{1}{2} z_i^2$ and using the reduced system (32), it is deduced that

$$\dot{V}_i \leq \left(\sum_{j=1}^{i} -c_j z_j^2\right) + z_i z_{i+1} - r\|e\|^2 + z_1^2 \quad (34)$$

Step n: In the last step, the actual control input u appears and is at our disposal. We derive the z_n dynamics

$$\dot{z}_n = \dot{\hat{x}}_n - \dot{\alpha}_{n-1} = k_n e_1 + f_n(\underline{\hat{x}}_n, u) + u - \dot{\alpha}_{n-1} \quad (35)$$

We now obtain an approximate actual control input via time-scale separation to satisfy

$$k_n e_1 + f_n(\underline{\hat{x}}_n, u) + u - \dot{\alpha}_{n-1} = -c_n z_n - z_{n-1} \quad (36)$$

as

$$\varepsilon_n \dot{u} = -sign\left(\frac{\partial Q_n}{\partial u}\right) Q_n(t, \bar{z}_n, u) \quad (37)$$

with the initial condition $u(0) = u_0$, $\varepsilon_n \ll 1$ and

$$\begin{aligned}Q_n(t, \bar{z}_n, u) &= c_n z_n + z_{n-1} + k_n e_1 \\ &+ f_n(\underline{\hat{x}}_n, u) + u - \dot{\alpha}_{n-1} \end{aligned} \quad (38)$$

where $\bar{z}_n = [z_1, \ldots, z_n]^T$ and c_n is the *n*th positive control gain. Let $u = h_n(t, \bar{z}_n)$ be an isolated root of $Q_n(t, \bar{z}_n, u) = 0$. Then the reduced system is defined as

$$\dot{z}_n = -c_n z_n - z_{n-1} \qquad z_n(0) = z_{n,0}, \tag{39}$$

and the boundary layer system can be represented by

$$\frac{dv_n}{d\tau_n} = -sign\left(\frac{\partial Q_n}{\partial u}\right) Q_n(t, \bar{z}_n, v_n + h_n(t, \bar{z}_n)) \tag{40}$$

where $v_n = u - h_n(t, \bar{z}_n)$and $\tau_n = t/\varepsilon_n$.

We choose the Lyapunov function $V_n = V_{n-1} + \frac{1}{2} z_n^2$. The resulting derivatives of V_n is given as

$$\dot{V}_n \leq -r\|e\|^2 - (c_1 - 1)z_1^2 - \sum_{j=2}^{n} c_j z_j^2 \tag{41}$$

Let $r > 0$, $\beta = min\{2r/\lambda_{min}(P),\ 2c_i\ , 2(c_1 - 1); i = 2, \ldots, n\}$

Then Eq. (41) becomes

$$\dot{V} \leq -\beta V \tag{42}$$

Equation (42) can be further rewritten as

$$V(t) \leq V(0)e^{-\beta t} \tag{43}$$

which guarantees the exponentially stability of the origin of reduced system (22), (32) and (39).

For the stability analysis of the proposed control system, we present the following theorem using Tikhonov's theorem (Theorem 11.2 in [17]).

Theorem 1: Consider the singular perturbation problem of the state observer (9) and the controllers (20), (30), (37). Assume that the following conditions are satisfied for all $[t, \bar{z}_{i+1}, \alpha_i - h_i(t, \bar{z}_{i+1}), \varepsilon_i] \in [0, \infty) \times D_{\bar{z}_{i+1}} \times D_{v_i} \times [0, \varepsilon_0)$ for some domains $D_{\bar{z}_{i+1}} \subset R^{i+1}$and $D_{v_i} \subset R$, which contain their respective origins, where $i = 1, \ldots, n$, $\bar{z}_{n+1} = \bar{z}_n$, $D_{\bar{z}_{n+1}} = D_{\bar{z}_n}$and $\alpha_n = u$.

$(B1)$ On any compact subset of $D_{\bar{z}_{i+1}} \times D_{v_i}$, the functions Q_i, their first partial derivatives with respect to $(\bar{z}_{i+1}, \alpha_i)$ and the first partial derivative of $Q_i$with respect to t are continuous and bounded. Also $h_i(t, \bar{z}_{i+1})$ and $(\partial Q_i/\partial \alpha_i)$have bounded first derivatives with respect to their arguments, $(\partial Q_i/\partial \bar{z}_{i+1})$ is Lipschitz in $\bar{z}_{i+1}$, uniform in *t*.

(B2) $(t, \bar{z}_{i+1}, v_i) \mapsto (\partial Q_i/\partial \alpha_i)(t, \bar{z}_{i+1}, v_i + h_i(t, \bar{z}_{i+1}))$ is bounded below by some positive constant for all $(t, \bar{z}_{i+1}) \in [0, \infty) \times D_{\bar{z}_{i+1}}$.

Then, the origins of (23), (33) and (40) are exponentially stable. Besides, let Ω_{v_i}be a compact subset of Γ_{v_i},where $\Gamma_{v_i} \subset D_{v_i}$, is the region of attraction of the autonomous system $(dv_i/d\tau_i) = -sign(\partial Q_i/\partial \alpha_i) Q_i(0, \bar{z}_{i+1,0}, v_i + h_i(0, \bar{z}_{i+1,0}))$with $\bar{z}_{i+1,0} = [z_{1,0}, \ldots, z_{i+1,0}]^T$.Then, for each compact subset $\Omega_{\bar{z}_n} \subset D_{\bar{z}_n}$, there exist a positive constant ε^* and $T > 0$ such that for all $t \geq 0$, $\bar{z}_{i+1,0} \in \Omega_{\bar{z}_{i+1}}$, $\alpha_{i,0} - h_i(0, \bar{z}_{i+1,0}) \in \Omega_{v_i}$ and $0 < \varepsilon < \varepsilon^*$, the system of (9), (20), (30) and (37) has the unique solution $\hat{x}_{i,\varepsilon_i}(t), i = 1, \ldots, n$ on $[0, \infty)$, and $\hat{x}_{1,\varepsilon_1}(t) = y_r(t) + O(\varepsilon_1)$ holds uniformly for $t \in [T, \infty)$.

Proof: For the use of Tikhonov's theorem, it should be verified that the conditions in our theorem satisfy assumptions (A1), (A2) and (A3) in Tikhonov's theorem. First, Assumption (B1) directly implies that Assumption (A1) holds. Second, we can show easily that Assumption (A2) holds because the origins of the reduced system (22), (32) and (39) are exponentially stable equilibrium points, that is, $\|\bar{z}_n(t)\| \leq \|\bar{z}_{n,0}\| e^{-w_0 t}$ for $t \geq 0$ and for some $w_0 > 0$ where $\bar{z}_{n,0} = [z_{1,0}, \ldots, z_{n,0}]^T$. From the converse Lyapunov theorem, there exists a Lyapunov function V_c such that

$$w_1\|\bar{z}_n\|^2 \leq V_c(t, \bar{z}_n) \leq w_2\|\bar{z}_n\|^2$$

$$\frac{\partial V_c}{\partial t}(t, \bar{z}_n) + \frac{\partial V_c}{\partial \bar{z}_n}(t, \bar{z}_n) C \bar{z}_n \leq -w_3\|\bar{z}_n\|^2 \tag{44}$$

where w_1, w_2, w_3 are positive constants and $C = \text{diag}[-c_1, \ldots, -c_n]$ denotes a diagonal matrix. We note that any positive c can be chosen in Assumption (A2), and so $\Omega_{\bar{z}_n} \subset \{\bar{z}_n \in D_{\bar{z}_n} | w_1(\bar{z}_n) \leq \rho c, 0 < \rho < 1\}$ can be any compact subset of $D_{\bar{z}_n}$.

Finally, we show that assumption (A3) holds. The exponential stability of the boundary layer system (23), (33) and (40) can be easily obtained locally by linearization with respect to v_i. Using Assumption 2 and (B2) yields

$$\text{sign}\left(\frac{\partial Q_i}{\partial \alpha_i}\right) = \text{sign}\left(\frac{\partial f_i}{\partial \alpha_i}\right) > 0 \tag{45}$$

$$\text{sign}\left(\frac{\partial Q_n}{\partial u}\right) = \text{sign}\left(\frac{\partial f_n}{\partial u}\right) > 0 \tag{46}$$

This implies that the boundary layer system has a locally exponentially stable origin. Therefore, we can apply Tikhonov's theorem. Accordingly, for each compact subset $\Omega_{\bar{z}_n} \subset D_{\bar{z}_n}$, there exist the constant $\varepsilon_i^* > 0$ and $T > 0$ such that for all $t \geq 0$ and $\bar{z}_{i+1,0} \in \Omega_{\bar{z}_{i+1}}$, $\alpha_{i,0} - h_i(0, \bar{z}_{i+1,0}) \in \Omega_{v_i}$ and $0 < \varepsilon < \varepsilon^*$, the system of (9), (20), (30) and (37) has the unique solution $\hat{x}_{i,\varepsilon_i}(t)$, $i = 1, \ldots, n$ on $[0, \infty)$, $\hat{x}_{1,\varepsilon_1}(t) = y_r(t) + O(\varepsilon_1)$ holds uniformly for $t \in [T, \infty)$.

5 Simulation Results

In this section, a simulation example is presented to show effectiveness of the proposed control approach. To compare our proposed approach with the designed control in [15], the following system already discussed in [15] is considered

$$\dot{x}_1 = x_1 + x_2 + \frac{x_2^3}{2}$$

$$\dot{x}_2 = x_1 x_2 + u + \frac{u^3}{7}$$

$$y = x_1 \tag{47}$$

The control object is to synthesize a control law u for pure feedback system (47) such that the output of system (47) follows the desired reference trajectory y_r generated from the van der Pol oscillator described by

$$\dot{x}_{d1} = x_{d2}$$

$$\dot{x}_{d2} = -x_{d1} + \beta(1 - x_{d1}^2)x_{d2}$$

$$y_r = x_{d1} \tag{48}$$

which yields a limit cycle trajectory when $\beta > 0$ ($\beta = 0.2$ in the simulation) for initial states starting from points other than $(0, 0)$. Let $x_{d1}(0) = 0.5$ and $x_{d2}(0) = 0.8$. Note that x_2 is immeasurable. The initial conditions are chosen as $x_1(0) = 0.5, x_2(0) = 0, \hat{x}_1(0) = 0, \hat{x}_2(0) = 0$, and the design parameters for the proposed control system are adopted as follows: $k_1 = 5, k_2 = 6, \varepsilon_1 = \varepsilon_2 = 0.01$. The simulation results are shown by Figures 1–4.

From Fig. 1, we can see that fairly good tracking performance is obtained. After a short transient process the output tracks the reference input at a high precision. Figures 2 and 3 demonstrate the prediction results between the system states and state estimation. Figure 4 shows the control input.These figures reveal that the proposed approach has the good control and prediction performance regardless of immeasurable states and non-affine property of the system. In addition, note that the states and the control input in the controlled closed-loop system are bounded.

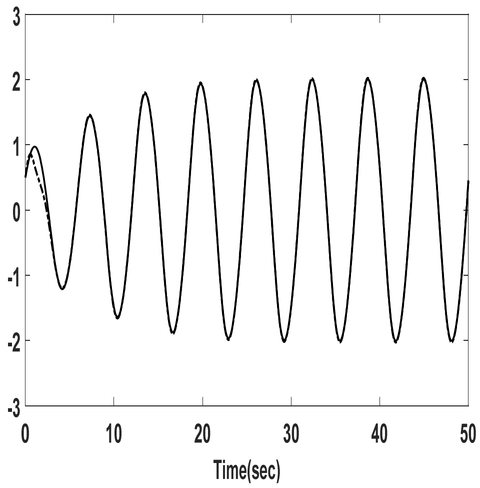

Fig. 1 x_1 (dash-dotted) and y_r (solid line).

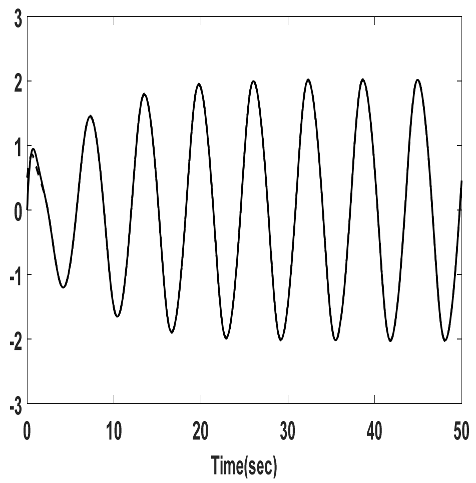

Fig. 2 x_1 (dash-dotted) and $\hat{x}_1$ (solid line).

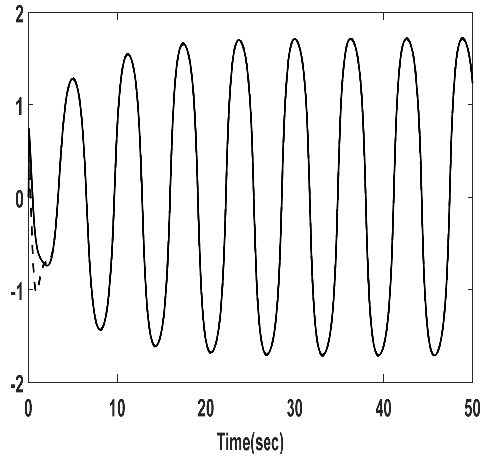

Fig. 3 x_2 (dash-dotted) and $\hat{x}_2$ (solid line).

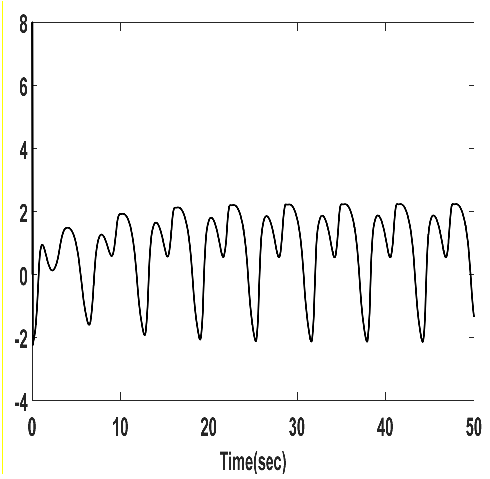

Fig. 4 Control input u.

In this part, control performances between the proposed approach in this paper and the two mentioned cases in [15] are compared. For this purpose, define the performance indexes of the observer errors as $I_1 = \sum_{k=1}^{n} |x_1(k) - \hat{x}_1(k)|$ and $I_2 = \sum_{k=1}^{n} |x_2(k) - \hat{x}_2(k)|$ .The tracking error and control indexes are defined as $I_3 = \sum_{k=1}^{n} |y(k) - y_r(k)|$ and $I_4 = \sum_{k=1}^{n} |u(k)|$, where n is the number of sampling data. The tracking error, observer errors and control indexes are calculated from 0 to 50s with a sampling period of 1s. From results discussion for two cases in [15], it was concluded that the faster convergence rates of the tracking and the observer errors are, the larger control energy is. However, according to Table 1, in our proposed approach all of indexes are smaller than in case 1. In the other words, it is not necessary to cost larger control energy to have smaller tracking error and observer errors. Therefore, in comparison to [15] with lower complexity and fewer design parameters, the reasonable results are obtained in this paper. Furthermore, the implementation of the controller is much simpler than in [15].

Table 1 Performance comparisons between proposed approach in this paper and the two cases in [15] with the tracking error, observer errors and control indexes.

Performance comparisons	**Case 1[15]**	**Case2 [15]**	**Proposed approach**
I_1	2.141	15.632	0.672
I_2	3.375	12.589	0.467
I_3	1.811	2.070	0.669
I_4	81.342	41.132	66.397

6 Conclusion

In this paper, an output feedback control approach has been developed for completely non-affine pure-feedback systems with immeasurable states. A state observer has been designed to estimate the immeasurable states. The backstepping and singular perturbation concept has been combined to design the virtual / actual control laws. The proposed control approach can overcome the non-affine property of pure-feedback systems with lower complexity and fewer design parameters. The stability proof is carried out by Tikhonov's theorem in singular perturbation theory. The simulation results show the good control and tracking performance of the system regardless of immeasurable states.

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